A FOURTH-ORDER MODIFICATION OF NEWTONS METHOD FOR MULTIPLE ROOTS

Zhinan Wu[1] & Xiaowu Li[2]
[1]School of Mathematics and Computer Science,Yichun University,Yichun 336000,PR China
[2]Computer and Information Engineerring College,Guizhou University for Nationalities,Guiyang 550025,PR China

ABSTRACT
In this paper, we present a new fourth-order modification of Newton's method for multiple roots, which is based on existing second -order. Modification of Newton's multiple root-finding methods. Some numerical examples illustrate that the new method is more efficient and performs better than other methods for multiple roots.

Keywords: *Multiple roots; nonlinear equations; Iterative methods; Newton's method; Order of convergence.*

1. INTRODUCTION

In this paper, we apply iterative method to find a zero α of multiplicity m of nonlinear equation $f(x) = 0$,where $f: D \subset R \to R$ is a scalar function on an open interval D. It is well known that Newton's method is one of the best iterative methods for solving a single nonlinear equation by using

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} \tag{1}$$

Which converges quadratically in some neighbourhood of α .And this method is only first-order convergence for solving a multiple root α of a nonlinear equation, the modified Newton's is

$$x_{n+1} = G(x_n) = x_n - m\frac{f(x_n)}{f'(x_n)} \tag{2}$$

Which converges quadratically in some neighbourhood of α .Eldon Hansen and Merrell Patrick[1] presented a iterative method Eldon Hansen and Merrell Patrick

$$x_{n+1} = x_n - \frac{m(m\beta+1)f(x_n)}{m\beta f'(x_n) \pm [m(m\beta-\beta+1)f'(x_n)^2 - m(m\beta+1)f(x_n)f''(x_n)]^{1/2}} \tag{3}$$

Which converges cubically to a root of multiplicity m for finite constant β .The special case of method (3) is Halley's method[2]

$$f_8(x) = (\sqrt{x^2+2x+5} - 2\sin(x) - x^2 + 3)^7, x^* = 2.3319676558839640$$

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{m+1}{2m}f'(x_n) - \frac{f(x_n)f''(x_n)}{2f'(x_n)}} \tag{4}$$

On the other hand, third-order method was developed by Victory and Neta [3] and based on King's four order method (for simple roots)[4]

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)} \\ x_{n+1} = y_n - \dfrac{f(y_n)}{f'(x_n)}\dfrac{f(x_n)+Af(y_n)}{f(x_n)+Bf(y_n)} \end{cases} \tag{5}$$

Where $A = (\frac{m}{m-1})^{2m} - (\frac{m}{m-1})^{m+1}$, $B = \dfrac{(\frac{m}{m-1})^m(m-2)(m-1)+1}{(m-1)^2}$.

Dong[5] has presented two third-order methods requiring two evaluations of f and one evaluation of f'

$$\begin{cases} y_n = x_n - \sqrt{m}u_n \\ x_{n+1} = y_n - m(1-\frac{1}{\sqrt{m}})^{1-m}\frac{f(y_n)}{f'(x_n)} \end{cases} \quad (6)$$

$$\begin{cases} y_n = x_n - u_n \\ x_{n+1} = y_n + \dfrac{u_n f(y_n)}{f(y_n)-(1-\frac{1}{m})^{m-1}f(x_n)} \end{cases} \quad (7)$$

Where $u_n = \dfrac{f(x_n)}{f'(x_n)}$

Two other third order methods developed by Dong [6], both require the same information and both based on a family of fourth order methods (for simple roots) due to Jarratt [7]:

$$\begin{cases} y_n = x_n - u_n \\ x_{n+1} = y_n - \dfrac{f(x_n)}{(\frac{m}{m-1})^{m+1}f'(y_n)+\frac{m-m^2-1}{(m-1)^2}f'(x_n)} \end{cases} \quad (8)$$

$$\begin{cases} y_n = x_n - \dfrac{m}{m+1}u_n \\ x_{n+1} = y_n - \dfrac{\frac{m}{m+1}f(x_n)}{(1+\frac{1}{m})^m f'(y_n) - f'(x_n)} \end{cases} \quad (9)$$

Osada [8] has developed a third-order method using the second derivative

$$x_{n+1} = x_n - \frac{1}{2}m(m+1)u_n + \frac{1}{2}(m-1)^2\frac{f'(x_n)}{f''(x_n)} \quad (10)$$

Where $u_n = \dfrac{f(x_n)}{f'(x_n)}$.

Changbum Chun and Beny Neta[9] developed a third-order method using the first and second derivative

$$x_{n+1} = x_n - \frac{2m^2 f(x_n)^2 f''(x_n)}{m(3-m)f(x_n)f'(x_n)f''(x_n)+(m-1)^2 f'(x_n)^3}. \quad (11)$$

B. Neta[10] given 3 new third-order method not requiring second derivative for finite constant θ, β, γ

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}[\beta + \gamma\frac{f(y_n)}{f(x_n)}], \quad (12)$$

$$x_{n+1} = x_n + \frac{m(m-3)}{2}u_n[1-\frac{m}{m-3}u_n w(x_n)], \quad (13)$$

$$x_{n+1} = x_n - \frac{m(m+1)}{2}u_n + \frac{(m-1)^2}{2w(x_n)}, \quad (14)$$

Where $y_n = x_n - \theta\dfrac{f(x_n)}{f'(x_n)}$, $w(x_n) = \dfrac{6(f(x_{n-1})-f(x_n))+2hf'(x_{n-1})+4hf'(x_n)}{h^2 f'(x_n)}$, $h = x_n - x_{n-1}$.

Changbum Chun, Hwa ju Bae and Beny Neta[11] suggested two one-parameter families of methods for multiple roots for finite constant θ

$$x_{n+1} = x_n - \frac{m[(2\theta-1)m+3-2\theta]}{2}\frac{f(x_n)}{f'(x_n)} + \frac{\theta(m-1)^2}{2}\frac{f'(x_n)}{f''(x_n)} - \frac{(1-\theta)m^2}{2}\frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3}, \quad (15)$$

$$\begin{cases} u_n = \dfrac{f(x_n)}{f'(x_n)} \\ y_n = x_n - u_n \\ x_{n+1} = y_n + \theta \dfrac{u_n f(y_n)}{f(y_n) - (1-\frac{1}{m})^{m-1} f(x_n)} - (1-\theta)\dfrac{f(y_n)}{f'(x_n)}\dfrac{f(x_n)+Af(y_n)}{f(x_n)+Bf(y_n)} \end{cases} \tag{16}$$

Where $A = \left(\dfrac{m}{m-1}\right)^{2m} - \left(\dfrac{m}{m-1}\right)^{m+1}, B = \dfrac{(\frac{m}{m-1})^m (m-2)(m-1)+1}{(m-1)^2}$.

Neta and Johnson [12] have developed a fourth order method requiring one function- and three derivative evaluation per step. The method is based on Jarratt's method [13] given by the iteration for finite constant a,b,c,a1,b1,c1

$$x_{n+1} = x_n - \frac{f(x_n)}{a_1 f'(x_n) + a_2 f'(y_n) + a_3 f'(\eta_n)}, \tag{17}$$

Where $u_n = \dfrac{f(x_n)}{f'(x_n)}$ and

$$\begin{aligned} y_n &= x_n - au_n \\ v_n &= \frac{f(x_n)}{f'(y_n)} \\ \eta_n &= x_n - bu_n - cv_n \end{aligned} \tag{18}$$

a,b,c are constants.

Neta [14] has developed a fourth-order method requiring one function- and three derivative-evaluation per step. The method is based on Murakami's method [15] given by the iteration for finite constants a1, a2, a3, b1, b2

$$x_{n+1} = x_n - a_1 u_n - a_2 v_n - a_3 w_3(x_n) - \psi(x_n), \tag{19}$$

Where $u_n = \dfrac{f(x_n)}{f'(x_n)}$, v_n, y_n and η_n are given by (18) and

$$\begin{aligned} w_3(x_n) &= \frac{f(x_n)}{f'(\eta_n)}, \\ \psi(x_n) &= \frac{f(x_n)}{b_1 f'(x_n) + b_2 f'(y_n)}. \end{aligned} \tag{20}$$

2. THE METHOD AND ANALYSIS OF CONVERGENCE

Now we rewrite (2), in the form

$$x_{n+1} = G(x_n) = x_n - m\frac{f(x_n)}{f'(x_n)} \tag{21}$$

By (21) , we can get

$$\begin{aligned} x_{n+1} &= G(G(x_n)) = G(x_n) - m\frac{f(G(x_n))}{f'(G(x_n))} \\ &= x_n - m\frac{f(x_n)}{f'(x_n)} - m\frac{f(x_n - m\frac{f(x_n)}{f'(x_n)})}{f'(x_n - m\frac{f(x_n)}{f'(x_n)})} \end{aligned} \tag{22}$$

From (22), we obtain

$$\begin{cases} y_n = x_n - m\dfrac{f(x_n)}{f'(x_n)} \\ x_{n+1} = y_n - m\dfrac{f(y_n)}{f'(y_n)} \end{cases} \qquad (23)$$

For (23),we have the following

Theorem 1. Let $\alpha \in I$ be a multiple root of multiplicity m of a sufficiently differentiable $f: I \to R$ for an open interval I. If x_0 is sufficiently close to α, suppose that order of convergence of iterative scheme $x_{n+1} = G(x_n)$ is k, then convergence rate of $x_{n+1} = \Phi(\Phi(x_n))$ is k^2.

Proof:

Because convergence rate of $x_{n+1} = G(x_n)$ is k. So error equation of $x_{n+1} = \Phi(\Phi(x_n))$ can be written as

$$e_{n+1} = K_0 e_n^{*k}, \qquad (24)$$

Where

$$e_n^* = K_1 e_n^k \qquad (25)$$

Substituting (25) into (24) , the error equation of $x_{n+1} = \Phi(\Phi(x_n))$ is

$e_{n+1} = K_2 e_n^{k^2}$, where K_0, K_1, K_2 are constants. The proof is completed.

Theorem 2. let $\alpha \in I$ be a multiple root of multiplicity m of a sufficiently differentiable $f: I \to R$ for an open interval I. If x_0 is sufficiently close to α, then the method defined by (23) has fourth-order convergence.

Proof: Because order of convergence of method (21) is 2, from theorem 1, we can know that the method defined by (23) has fourth-order convergence. The Proof is completed.

3. NUMERICAL EXAMPLES

Table 1 Comparison of various multiple-roots iterative methods.

f(x)	NM	HM	OM	ECM	CM	NEM	CHM	Chang	NM2
Parameter(s)							θ=0.5,-1	θ=-1,0.1	
$f_1, x_0 = 5.0$	20	15	969	2277	81	244	609.540	1008,34,969	20
$f_2, x_0 = 5.0$	12	516	21	NC	24	100	NC,NC	15,NC,18	12
$f_3, x_0 = 5.0$	12	12	12	12	12	116	222,NC	12,12,12	12
$f_4, x_0 = -2.0$	28	12	111	12	12	80	330,NC	414,12,108	18
$f_5, x_0 = 5.0$	14	18	15	15	15	148	477,NC	15,15,15	12
$f_6, x_0 = 4.5$	44	15	33	NC	NC	NC	567,NC	NC,NC,33	44
$f_7, x_0 = 5.0$	8	15	15	15	15	15	468,NC	15,18,15	8
$f_8, x_0 = 5.0$	12	12	15	12	12	12	975,NC	12,12,15	12

We employ the method(23)(NM2) suggested in this paper to solve multiple roots for nonlinear equation and compare them with the Newton method(2)(NM),Halley-like method(4)(HM),Osaka's method(10)(OM),the Euler_Chebyshev method((19)(ECM)[9],[16])),the method(11)(CM),the method(17)(NEM),the method(16)(CHM),the method(15)(Chang).All computations were done using VC++ 6.0 with 17 digit floating point arithmetic. Displayed in Table 1 are the number of function evaluations (NFE) required such that $|f(x_n)|$<1.E-

17, $|x_n - x^*|$ <1.E-17.NC in Table 1 means that the method does not converge to the root. In Table 1,we user the following test functions and display the approximate zeros x^* found up to the 17th decimal place:

$$f_1(x) = (x^5 + 2x^4 - 4x^2 - 17)^4, x^* = 1.663749178672671$$

$$f_2(x) = (\sin^2(x) - 2x + 1)^2, x^* = 0.71483582544138924$$

$$f_3(x) = (8xe^{-x^2} - 2x - 3)^2, x^* = -1.7903531791589544$$

$$f_4(x) = (2x\cos(x) + x - 3)^3, x^* = -3.0346643069740450$$

$$f_5(x) = (e^{-x^2+x+3} - x + 2)^3, x^* = 2.4905398276083051$$

$$f_6(x) = (e^{-x} + 2\sin(x))^4, x^* = 3.1627488709263654$$

$$f_7(x) = (\ln(x^2 + 3x + 5) - 2x + 7)^3, x^* = 5.4690123359101421$$

$$f_8(x) = (\sqrt{x^2 + 2x + 5} - 2\sin(x) - x^2 + 3)^7, x^* = 2.3319676558839640$$

For example, it can be seen that the method(ECM), the method(CM), the method(NEM), the method(CHM), the method(Chang) have sensitivities to the original value or to the parameter of iterative method, or multiplicity m of the multiple roots. The computational results in Table 1 show that the method (22)(NM2) require less NFE than NM,HM,OM,ECM,CM,NEM,CHM and Chang. The method (22)(NM2) has iteration stability to the original iteration value and multiplicity m of the multiple roots. Therefore, they are of practical interest and can compete with other methods.

4. CONCLUSION

Based on the modified Newton's iterative method, we give further modification of the Newton's method to obtain a fourth-order convergence iterative method. Several examples show that the new method presented in the paper is more efficient and perform better than the modified Newton's method and some other methods.

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