THE COMPARISON OF HAMILTON METHOD WITH RAYLEIGH'S AND RAYLEIGH-RITZ METHODS FOR THE NATURAL FREQUENCIES OF THREE-SUPPORTED BEAM

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ABSTRACT

In this study, three supported beam was studied. Longitudinal and transverse vibrations were investigated. Natural frequencies were calculated using Hamilton, Rayleigh's and Rayleigh-Ritz Methods. For calculations, functions providing all boundary conditions of the problem were suggested. As a result natural frequencies of the system without using the equations of motion were calculated. Obtained results were compared with the real values.

Keywords: *Natural frequency, Vibration, Rayleigh's method, Rayleigh-Ritz method*

1. INTRODUCTION

Linear first natural vibrations of three supported beams are studied in detail. Beam-support systems are frequently used as design models in engineering. Srinivasan [4] investigated free vibrations of stretched beams using Ritz-Galerkin method. Tseng and Dugundji [5] studied vibrations of beams in buckling analytically and experimentally. Dowell [6] analyzed free vibrations of a simply supported beam and beam-spring system using approximate solution techniques. Szemplinska-Stupnicka [7] considered nonlinear end conditions case using generalized Ritz method. Özkaya [9] presented the effects of different end conditions for beam-mass systems. More recent works on this type are due to [10-22]. For slightly curved beams with stretching, one may refer to Rehfield [23] and *Öz et al*.[24]. Nonlinear vibrations and 3:1 internal resonances on multiple supports were investigated and excitation frequency-frequency response curves drawn for different support numbers [25].

In this study, nonlinear transverse vibrations of an Euler – Bernoulli beam with simple supports is considered. The beam stretches during vibration due to immovable supports. Equations of motion are solved for 3 support cases by using three different methods. First natural frequencies are calculated and then the results will be compared.

2. HAMILTON METHOD

The equations of motion are obtained for three support case. Rotary inertia and shear effect are not included and cross sectional area do not change during motion. Location of the intermediate support is shown by x_s^* and L is the total length in Fig. 1.

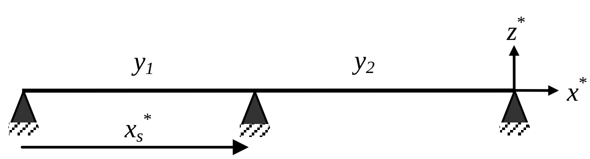

Figure 1 Three simple supported beam

2.1 Longitudinal Vibration

The Lagrangian can be written as follows,

$$\mathcal{L} = T - V \tag{1}$$

$$\mathcal{L} = \frac{1}{2}\int_0^{x_s} \rho A \left(\frac{\partial y_1}{\partial t}\right)^2 dx + \frac{1}{2}\int_{x_s}^{L} \rho A \left(\frac{\partial y_2}{\partial t}\right)^2 dx - \frac{1}{2}\int_0^{x_s} EA \left(\frac{\partial y_1}{\partial x}\right)^2 dx - \frac{1}{2}\int_{x_s}^{L} EA \left(\frac{\partial y_2}{\partial x}\right)^2 dx \tag{2}$$

$$\delta \int_{t_1}^{t_2} \mathcal{L}\, dt = 0 \tag{3}$$

The last two integrals denote energies due to tension force similarly. ρA is the mass per unit length, EA is longitudinal rigidity, EI is flexural rigidity. Hamilton principle can be applied to that equation and the equations of motion can be obtained as follows. System's movement equations are given below

$$-\rho A\frac{\partial^2 y_1}{\partial t^2}+EA\frac{\partial^2 y_1}{\partial x^2}=0\rightarrow \rho A\frac{\partial^2 y_1}{\partial t^2}=EA\frac{\partial^2 y_1}{\partial x^2} \quad (4)$$

$$-\rho A\frac{\partial^2 y_2}{\partial t^2}+EA\frac{\partial^2 y_2}{\partial x^2}=0\rightarrow \rho A\frac{\partial^2 y_2}{\partial t^2}=EA\frac{\partial^2 y_2}{\partial x^2} \quad (5)$$

Boundary conditions are given below

$Y_1(0) = 0$, $Y_1(x_s) = Y_2(x_s)$, $Y_1'(x_s) = Y_2'(x_s)$, $Y_2(L) = 0$

The natural frequencies are [29] given below

$$w_1 = 1{,}5708\sqrt{\frac{EA}{mL^2}} \quad (6)$$

$$w_2 = 3{,}14159\sqrt{\frac{EA}{mL^2}} \quad (7)$$

2.2 Transverse Vibration

The Lagrangian can be written as follows

$$\mathcal{L}=T-V \quad (8)$$

$$\mathcal{L}=\frac{1}{2}\int_0^{x_s}\rho A\left(\frac{\partial y_1}{\partial t}\right)^2dx+\frac{1}{2}\int_{x_s}^{L}\rho A\left(\frac{\partial y_2}{\partial t}\right)^2dx-\frac{1}{2}\int_0^{x_s}EI\left(\frac{\partial^2 y_1}{\partial x^2}\right)^2dx-\frac{1}{2}\int_{x_s}^{L}EI\left(\frac{\partial^2 y_2}{\partial x^2}\right)^2dx \quad (9)$$

$$\delta\int_{t_1}^{t_2}\mathcal{L}\,dt=0 \quad (10)$$

After the calculation of Eq.(10). Equations of the motions are below

$$\rho A\frac{\partial^2 y_1}{\partial t^2}+EI\frac{\partial^4 y_1}{\partial x^4}=0 \quad (11)$$

$$\rho A\frac{\partial^2 y_2}{\partial t^2}+EI\frac{\partial^4 y_2}{\partial x^4}=0 \quad (12)$$

Boundary conditions are given below

$Y_1(0) = 0$, $Y_1''(0) = 0$, $Y_1(x_S) = 0$, $Y_1'(x_S) = Y_2'(x_S)$, $Y_1''(x_S) = Y_2''(x_S)$,

$Y_2(x_S) = 0$, $Y_2(L) = 0$, $Y_2''(L) = 0$ (13)

The natural frequencies are [32] given below

$$w_1 = 3{,}14159\sqrt{\frac{EI}{mL^2}} \quad (14)$$

$$w_2 = 25{,}13272\sqrt{\frac{EI}{mL^2}} \quad (15)$$

3. RAYLEIGH AND RAYLEIGH-RITZ METHOD

Without calculating movement equations, Rayleigh and Rayleigh-Ritz methods are generally used to find out fundamental frequency. The method can be used for both continuous and separated systems. Let's consider m_i physical point of a separated system. The transposition of physical point can be shown with matrix notation.

$$\{q(t)\} = \{u\}\, f(t) \quad (16)$$

Here f(t), is a harmonic function with ω frequency. The maximum value of kinetic energy

$$T_{max} = \frac{1}{2}\{u\}^t[m]\{u\}\omega^2 \quad (17)$$

This expression in terms of reference kinetic energy,

$$T^* = \frac{1}{2}\{u\}^t[m]\{u\} \quad (18)$$

In this case, maximum kinetic energy is obtained as below,

$$T_{max} = T^*\omega^2 \quad (19)$$

Similarly, maximum potential energy is written,

$$V_{max} = \frac{1}{2}\{u\}^t[k]\{u\} \quad (20)$$

If T_{max} and V_{max} values represented with in Eq.(19) and Eq. (20) are equaled each other and expression is arranged,

$$\omega^2 = \frac{V_{max}}{T^*} \quad (21)$$

Equation (21) is acquired. This statement gives Rayleigh rate. The natural frequency that is acquired with Rayleigh method is always higher than the real frequency. How the estimated mode structure is close to real, calculated natural frequency is close to real. Equation is possible only when the own function is used.

Unfortunately it is impossible to find out the solution of own value problem about continuous system which has non-uniform mass and rigidity dispersion. The Rayleigh rate correlation written for separated systems in Eq.(22) is also valid for continuous systems. Because, in the statement there are maximum value of potential energy and reference kinetic energy. But because of the impossibility for giving general expressions for continuous systems like separated systems, special formulation for every problem should be specified. In Rayleigh-Ritz method, there is an acceptance below related to solution function for any kind of continuous system:

$$U_{(x)} = \sum_{i=1}^{n} a_i\, u_{(x)_i} \quad (22)$$

As it is seen, a serial consist of n terms was chosen for solution function. In this serial; $u_{(x)}$'s functions related to x; a_i's are coefficients (Eigen Vector) that distinguish modes. As it is known, continuous systems have degrees of freedom. How many terms are used, the modes is an error rate for the self-worth and self-cut as many modes can be found. To generate rigidity and mass matrix, these expressions are used,

$$\frac{\partial V_{max}}{\partial a_i} = \sum_{j=1}^{n} k_{ij}\, a_j \quad (23)$$

$$\frac{\partial T^*}{\partial a_i} = \sum_{j=1}^{n} m_{ij}\, a_j \quad (24)$$

If problem of Eigen value is written like a separated system,

$$\begin{bmatrix} k_{11} & k_{21} \\ k_{12} & k_{22} \end{bmatrix} . \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \omega^2 \begin{bmatrix} m_{11} & m_{21} \\ m_{12} & m_{22} \end{bmatrix} . \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \quad (25)$$

equation is obtained. By equalizing determinant of coefficients to zero in Eq.(25), forth degree equation related to w is got:

$$\begin{bmatrix} k_{11}-\omega^2 m_{11} & k_{12}-\omega^2 m_{12} \\ k_{21}-\omega^2 m_{21} & k_{22}-\omega^2 m_{22} \end{bmatrix} = 0 \tag{26}$$

From the roots of this equation approximate expressions of natural frequency are acquired. If it is wanted to decrease mistake rates for first and second modes, a series with three or more terms must be written. But in this case, calculations will be more complicated.

3.1. Longitudinal and Transverse Vibration

The notations used in longitudinal vibrations are these,

V_{max} and T* can be written below

$$V_{max} = \frac{1}{2}\int_0^L EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx \tag{27}$$

$$T^* = \frac{1}{2}\int_0^L m(x)U^2(x)dx \tag{28}$$

If these expressions are replaced in Eq. (21),

$$\omega^2 = \frac{V_{max}}{T^*} = \frac{\int_0^L EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx}{\int_0^L m(x)U^2(x)dx} \tag{29}$$

is obtained.

V_{max} and T^* equations of transverse vibration are as follows,

$$V_{max} = \frac{1}{2}\int_0^L EI(x)\left[\frac{d^2U(x)}{dx^2}\right]^2 dx \tag{30}$$

$$T^* = \frac{1}{2}\int_0^L m(x)U^2(x)dx \tag{31}$$

If these expressions are replaced in Eq. (21),

$$\omega^2 = \frac{V_{max}}{T^*} = \frac{\int_0^L EI(x)\left[\frac{d^2U(x)}{dx^2}\right]^2 dx}{\int_0^L m(x)U^2(x)dx} \tag{32}$$

is obtained.

3.2. Longitudinal Vibrations

The system is being analyzed about this study in Fig 1.

3.2.1. Rayleigh Energy Method

Comparison function must provide all boundary conditions. Boundary conditions are as follows,

$$\mathrm{U}(0) = 0, \qquad \frac{dU(L)}{dx} = 0 \tag{33}$$

Comparison Function is given as,

$$U(x) = c_0\left[\left(\frac{x}{L}\right)^5 - \frac{5x}{L}\right] \tag{34}$$

$$\omega^2 = \frac{V_{max}}{T^*} = \frac{\int_0^L EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx}{\int_0^L m(x)U^2(x)dx} \tag{35}$$

Frequency formula was acquired by using in Eq. (35).

$$\omega^2 = \frac{V_{max}}{T^*} = \frac{\int_0^{x_s} EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx + \int_{x_s}^L EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx}{\int_0^{x_s} m(x)U^2(x)dx + \int_{x_s}^L m(x)U^2(x)dx} \tag{36}$$

$$U(x) = c_0\left[\left(\frac{x}{L}\right)^5 - \frac{5x}{L}\right] \tag{37}$$

Equation (34) was replaced in Eq. (35) and following integrals were solved.

$$\int_0^{x_s} EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx = \int_0^{x_s} EA(x)\left[\frac{d\left(c_0\left[\left(\frac{x}{L}\right)^5 - \frac{5x}{L}\right]\right)}{dx}\right]^2 dx$$
$$= 25EA{c_0}^2 \int_0^{x_s}\left(\frac{x^4}{L^5} - \frac{1}{L}\right)^2 dx = 12{,}1929\,\frac{EA{c_0}^2}{L} \tag{38}$$

$$\int_{x_s}^L EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx = \int_{x_s}^L EA(x)\left[\frac{d\left(c_0\left[\left(\frac{x}{L}\right)^5 - \frac{5x}{L}\right]\right)}{dx}\right]^2 dx$$
$$= 25EA{c_0}^2 \int_{x_s}^L\left(\frac{x^4}{L^5} - \frac{1}{L}\right)^2 dx = 5{,}5848\,\frac{EA{c_0}^2}{L} \tag{39}$$

$$\int_0^{x_s} m(x)U^2(x)dx = \int_0^{x_s} m(x)(c_0\left[\left(\frac{x}{L}\right)^5 - \frac{5x}{L}\right])^2 dx = 1{,}0305m{c_0}^2 L \tag{40}$$

$$\int_{x_s}^L m(x)U^2(x)dx = \int_{x_s}^L m(x)(c_0\left[\left(\frac{x}{L}\right)^5 - \frac{5x}{L}\right])^2 dx = 5{,}9651m{c_0}^2 L \tag{41}$$

Acquired values replaced in equation (35) and,

$$\omega^2 = \frac{V_{max}}{T^*} = \frac{\int_0^{x_s} EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx + \int_{x_s}^L EA(x)\left[\frac{dU(x)}{dx}\right]^2 dx}{\int_0^{x_s} m(x)U^2(x)dx + \int_{x_s}^L m(x)U^2(x)dx} = 2{,}5412\,\frac{EA}{mL^2} \tag{42}$$

approximate value of natural frequency in Rayleigh method was obtained.

$$\omega = 1{,}5941\sqrt{\frac{EA}{mL^2}} \tag{43}$$

3.2.2. Rayleigh-Ritz Method

Comparison function is given as

$$U(x) = a_1\left[\left(\frac{x}{L}\right)^3 - \frac{3x}{L}\right] - a_2[x^8 - 4L^6x^2] \tag{44}$$

Comparison function allows for all boundary conditions of the system. Depending on comparison function, maximum potential energy and kinetic energy equations are given as,

$$V_{max} = \int_0^{x_s} EA \left[\frac{dU(x)}{dx}\right]^2 dx + \int_{x_s}^{L} EA \left[\frac{dU(x)}{dx}\right]^2 dx \quad (45)$$

$$T^* = \int_0^{x_s} m(x) U^2(x) dx + \int_{x_s}^{L} m(x) U^2(x) dx \quad (46)$$

Comparison function was replaced in equations and the values of maximum potential energy and kinetic energy are given follows,

$$V_{max} = (2{,}4\frac{{a_1}^2}{L} + 5{,}4a_1a_2L^7 + 5{,}6888{a_2}^2L^{15})EA \quad (47)$$

$$T^* = (0{,}9714{a_1}^2L + 2{,}1166a_1a_2L^9 + 1{,}2657{a_2}^2L^{17})m \quad (48)$$

Rigidity and mass matrix were made up below

$$\frac{\partial V_{max}}{\partial a_1} = (\frac{4{,}8}{L}a_1 + 5{,}4a_2L^7)EA \quad (49)$$

$$\frac{\partial V_{max}}{\partial a_2} = (5{,}4a_1L^7 + 11{,}3776a_2L^{15})EA \quad (50)$$

$$k_{11} = 4{,}8\frac{EA}{L}\ ; \quad k_{12} = k_{21} = 5{,}4L^7EA\ ; \quad k_{22} = 11{,}3776L^{15}EA \quad (51)$$

$$\frac{\partial T^*}{\partial a_1} = (1{,}9428a_1L + 2{,}1166a_2L^9)m \quad (52)$$

$$\frac{\partial T^*}{\partial a_1} = (2{,}1166a_1L^9 + 2{,}5314a_2L^{17})m \quad (53)$$

$$m_{11} = 1{,}9428L\,m; \quad m_{12} = m_{21} = 2{,}1166L^9m; \quad m_{22} = 2{,}5314L^{17}m \quad (54)$$

Approximate expressions of natural frequency were obtained from the following equation.

$$\begin{vmatrix} k_{11} - \omega^2 m_{11} & k_{12} - \omega^2 m_{12} \\ k_{21} - \omega^2 m_{21} & k_{22} - \omega^2 m_{22} \end{vmatrix} = 0 \quad (55)$$

The frequency of the first and second mode is as follows.

$$\omega_1 = 1{,}570830\sqrt{\frac{EA}{mL^2}} \quad (56)$$

$$\omega_2 = 4.85291\sqrt{\frac{EA}{mL^2}} \quad (57)$$

3.3. Transverse Vibrations

3.3.1. Rayleigh Energy Method

Comparison function is given as follows,

$$U(x) = c_1\left[\frac{3}{2}L^4x^2 - \frac{2}{3}L^3x^3 + \frac{1}{30}x^6\right] \tag{58}$$

Comparison function allows for all boundary conditions of the system.

$$\omega^2 = \frac{V_{max}}{T^*} = \frac{\int_0^L EI(x)\left[\frac{d^2U(x)}{dx^2}\right]^2 dx}{\int_0^L m(x)U^2(x)dx} \tag{59}$$

$$\int_0^{x_s} EI\left[\frac{d^2U(x)}{dx^2}\right]^2 dx = EI{c_1}^2\int_0^{x_s}[3L^4 - 4L^3x + x^4]^2 dx = 2{,}1835EI{c_1}^2L^9 \tag{60}$$

$$\int_{x_s}^L EI\left[\frac{d^2U(x)}{dx^2}\right]^2 dx = EI{c_1}^2\int_{x_s}^L[3L^4 - 4L^3x + x^4]^2 dx = 0{,}1275EI{c_1}^2L^9 \tag{61}$$

$$\int_0^L mU^2(x)dx = m{c_1}^2\int_0^{x_s}\left[\frac{3}{2}L^4x^2 - \frac{2}{3}L^3x^3 + \frac{1}{30}x^6\right]^2 dx = 0{,}0093\, m{c_1}^2L^{13} \tag{62}$$

$$\int_{x_s}^{x_2} mU^2(x)dx = m{c_1}^2\int_{x_s}^L\left[\frac{3}{2}L^4x^2 - \frac{2}{3}L^3x^3 + \frac{1}{30}x^6\right]^2 dx = 0{,}1775\, m{c_1}^2L^{13} \tag{63}$$

Calculated results were replaced in equation (32) and

$$\omega^2 = \frac{V_{max}}{T^*} = \frac{\int_0^{x_s} EI\left[\frac{d^2U(x)}{dx^2}\right]^2 dx + \int_{x_s}^L EI\left[\frac{d^2U(x)}{dx^2}\right]^2 dx}{\int_0^{x_s} mU^2(x)dx + \int_{x_s}^L mU^2(x)dx} = 12{,}3715\frac{EI}{mL^4} \tag{64}$$

was obtained.

Approximate value of natural frequency in Rayleigh method is

$$\omega = 3{,}5173\sqrt{\frac{\mathrm{EI}}{\mathrm{mL}^4}} \tag{65}$$

3.3.2. Rayleigh-Ritz Method

To find out approximate values of first and second own frequencies of transverse vibration, comparison function was suggested. Comparison function:

$$U(x) = a_1\left(\frac{3}{2}L^4x^2 - \frac{2}{3}L^3x^3 + \frac{1}{30}x^6\right) + a_2\left(28L^6x^2 - \frac{28}{5}L^3x^5 + x^8\right) \tag{66}$$

Comparison function that will be used provides system's all boundary conditions. Depending on comparison function, maximum potential energy and reference kinetic energy equations are below:

$$V_{max} = \frac{1}{2}\int_0^{x_s} EI\left[\frac{d^2U(x)}{dx^2}\right]^2 dx + \frac{1}{2}\int_{x_s}^L EI\left[\frac{d^2U(x)}{dx^2}\right]^2 dx = \left(1{,}1555L^9{a_1}^2 + 59{,}8909L^{11}a_1a_2 + 837{,}4153L^{13}{a_2}^2\right)EI \tag{67}$$

$$T^* = \int_0^{x_s} m(x)U^2(x)dx + \int_{x_s}^L m(x)U^2(x)dx =$$

$$(0,0934L^{13}a_1{}^2 + 4,8248L^{15}a_1a_2 + 62,4003L^{17}a_2{}^2)m \tag{68}$$

Rigidity and mass matrix were made up as following

$$\frac{\partial Vmax}{\partial a_1} = (2.311a_1L^9 + 59,8909L^{11}a_2)EI \tag{69}$$

$$\frac{\partial Vmax}{\partial a_2} = (59,8909L^{11}a_1 + 1674,8306a_2L^{13})EI \tag{70}$$

$$k_{11} = 2.311L^9EI; \quad k_{12} = k_{21} = 59,8909L^{11}EI\ ; \quad k_{22} = 1674,8306L^{13}EI \tag{71}$$

$$\frac{\partial T^*}{\partial a_1} = (0,1868a_1L^{13} + 4,8248L^{15}a_2)m \tag{72}$$

$$\frac{\partial T^*}{\partial a_2} = (4,8248L^{15}a_1 + 124,8006a_2L^{17})m \tag{73}$$

$$m_{11} = 0,1868L^{13}m; \quad m_{12} = m_{21} = 4,8248L^{15}m; \quad m_{22} = 124,8006L^{17}m \tag{74}$$

were acquired. Natural frequency's approximate values were obtained from the following equation below.

$$\begin{vmatrix} k_{11} - \omega^2 m_{11} & k_{12} - \omega^2 m_{12} \\ k_{21} - \omega^2 m_{21} & k_{22} - \omega^2 m_{22} \end{vmatrix} = 0; \tag{75}$$

$$\omega_1 = 3,51706\sqrt{\frac{E\,I}{m\,L^4}} \tag{76}$$

$$\omega_2 = 25,9466\sqrt{\frac{E\,I}{m\,L^4}} \tag{77}$$

Table 1. Longitudinal Vibrations for three different methods

	Hamilton Method	**Rayleigh Energy Method**	**Rayleigh-Ritz Method**
w₁	$1,5708\sqrt{\frac{EA}{mL^2}}$	$1,5941\sqrt{\frac{EA}{mL^2}}$	$1,570830\sqrt{\frac{EA}{mL^2}}$
w₂	$4,71238\sqrt{\frac{EA}{mL^2}}$		$4.85291\sqrt{\frac{EA}{mL^2}}$

Table 2. Transverse vibrations for three different methods

	Hamilton Method	**Rayleigh Energy Method**	**Rayleigh-Ritz Method**
w_1	$3,14159\sqrt{\frac{EI}{mL^4}}$	$3,5173\sqrt{\frac{EI}{mL^4}}$	$3,51706\sqrt{\frac{EI}{mL^4}}$
w_2	$25,13272\sqrt{\frac{EI}{mL^4}}$		$25,9466\sqrt{\frac{EI}{mL^4}}$

4. RESULTS AND DISCUSSION

In this study firstly problem was solved by Hamilton Method, then how Rayleigh and Rayleigh-Ritz rate is applied to separated and continuous systems was explained. A comparison function that provides all boundary conditions was offered while solving longitudinal and transverse vibration of three-supported beam. As a result, without solving movement equations, natural frequencies were calculated by using Rayleigh and Rayleigh-Ritz methods. Natural frequency values acquired with approximate methods are very close to true results which were got analytically. According to the comparison function's goodness, this familiarity increases more. But the results of this method are not single. For every time if better comparison functions are used, getting better results are possible. These first approach calculations are used to find natural frequencies of mechanic systems in engineering. These values can be obtained by analytical solutions of the problem. But this takes more time. Also, especially in continuous systems, it is impossible to get closed solution of lots of problems.

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