THE SOLUTION OF THE FRACTIONAL DIFFERENTIAL EQUATION WITH THE GENERALIZED TAYLOR COLLOCATIN METHOD

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ABSTRACT

In this paper, we propose the generalized Taylor collocation method for solving the variable coefficients fractional differential equation of order 2α for $\alpha \in (0,1]$ under the given initial or boundary conditions and give matrix representations of the problem. Additionally, analytical form solution of the problem is calculated by using this technique.

Keyword: *Fractional differential equation, Taylor collocation method, Collocation points.*

1. INTRODUCTION

In this paper, we consider the variable coefficients fractional differential equation of order 2α :

$$\left[\mathbf{L}_{2\alpha}(y)\right](x) := P_2(x)y^{(2\alpha)}(x) + P_1(x)y^{(\alpha)}(x) + P_0(x)y(x) = f(x) \tag{1}$$

where $\alpha \in (0,1]$, $P_2 \neq 0$, P_0, P_1 and f are the arbitrary functions of x defined on the interval [a,b] having nth order differentiation, $y^{(k\alpha)}$ for $k = 0,1,2$ denotes the $k\alpha.$ order fractional differentiation of y with respect to x such that α is a arbitrary number in the interval $(0,1]$. More than three hundreds years, the applications of the fractional calculus that are allowed the related problems to be more understandable, are improved and are extended in almost all fields of mathematics and the other sciences. Using the fractional differential equations modeled in many areas, the obtained constructions are needed to be solved. The fractional calculus is dealt by many authors in most cynosure fundamental books are dealt he fractional calculus is. For example, the fractional calculus on bioengineering in [1], the fundamental solution of the space–time fractional diffusion equation.

Recently, fractional calculus has found new applications in assorted fields, such as engineering, physics, finance, chemistry, bioengineering [13-15,19,21-23] etc. and is still used in new numerical simulations of the chaotic systems [5,24], real world applications [1], control processing [20]. Also the fractional variational principles have developed and applied to fractional problems [2]. During the last years, He's variational iteration method has extended to solve the fractional differential equations [6-8,11,12,16,17].

In the case in which α belongs to the interval $(0,1]$ by not changing the structure of the second order differential equations with variable coefficients (1), the approximate solutions are obtained. For this purpose, we will proceed the Taylor collocation method that is presented by Sezer and Karamete for general form of the

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) = f(x) \tag{2}$$

[25]. This method is introduced for solving integral equations by Kanwall ve Liu [28] and is developed by Sezer [29,30]. Recently, Çenesiz proposed a method in order to apply Bagley-Torvik fractional differential equation in [31].

In this study, we also propose Taylor collocation method in fractional sense to solve the differential equation (1) which has a fractional derivative.

2. PRELIMINARIES AND NOTATIONS

The definitions of fractional derivative are considered in many papers. In most recently [13] the definitions of Riemann-Liouville, Caputo fractional operators are given in sense of right and left side. The definition of Riemann-Liouville fractional integral operator and the Caputo fractional differential operator are used during our investigations.

Definition 2.1: Let $\Omega = [a,b]$ $(-\infty < a < b < \infty)$ be a finite interval on real axis R. The Riemann-Liouville fractional integral of order $\alpha \in R$ $(\alpha > 0)$ of a function $f \in C_\mu, \mu \geq -1$, $I_{a+}^{\alpha} f$

$$(I_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \quad (x > a; \alpha > 0) \tag{3}$$

and $(I_{a+}^{0} f)(x) = f(x)$ for $\alpha = 0$.

For this operator at most common properties are

i) $(I_{a+}^{\alpha} I_{a+}^{\beta} f)(x) = (I_{a+}^{\alpha+\beta} f)(x) \qquad (\alpha \geq 0, \beta \geq 0)$ (4)

ii) $(I_{a+}^{\alpha} I_{a+}^{\beta} f)(x) = (I_{a+}^{\beta} I_{a+}^{\alpha} f)(x) \qquad (\alpha \geq 0, \beta \geq 0)$ (5)

iii) $(I_{a+}^{\alpha} t^{\beta-1} f)(x) = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} x^{\beta+\alpha-1} \qquad (\alpha \geq 0, \beta > 0)$ (6)

Since the Riemann-Liouville definition for fractional derivative is unsuitable for initial value fractional problems, we shall give the definition of Caputo fractional derivative (as in [13]):

Definition 2.2: The fractional derivative $({}^{C}D_{a+}^{\alpha} f)(x)$ of order $\alpha \in R$ $(\alpha \geq 0)$ on $[a,b]$ is defined by

$$({}^{C}D_{a+}^{\alpha} f)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt = (I_{a+}^{n-\alpha} D^n f)(x) \tag{7}$$

where $D = \dfrac{d}{dx}$, $n = \lfloor \alpha \rfloor + 1$ $n \in N$ and $f(x) \in AC^n[a,b]$ in [13].

Lemma 2.3: If $n-1 < \alpha < n, n \in N$ and $f \in C_\mu^n, \mu \geq -1$, then

$$({}^{C}D_{a+}^{\alpha} I_{a+}^{\alpha} f)(x) = f(x) \tag{8}$$

and

$$(I_{a+}^{\alpha}\, {}^{C}D_{a+}^{\alpha} f)(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!} \qquad x > 0. \tag{9}$$

The reason of using Caputo fractional derivative is its ascendancy than other definitions of fractional derivatives in applying to traditional initial and boundary problems. For more information about fractional derivatives, integrals and theirs properties readers can consult to [4,9,13].

It is also used the following definition of Generalized Taylor's Formula that has already been written as a formal version in [6]:

Theorem 2.4: Suppose that $D_a^{k\alpha} f(x) \in C[a,b]$ for $k = 0,1,\ldots,n+1$ where $0 < \alpha \leq 1$, then we have the Taylor Series expansion about $x = \tau$

$$f(x) = \sum_{i=0}^{n} \frac{(x-\tau)^{i\alpha}}{\Gamma(i\alpha+1)} D_a^{i\alpha} f(\tau) + \frac{(D_a^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha+1)} (x-\tau)^{(n+1)\alpha} \tag{10}$$

with $a \leq \xi \leq x, \forall x \in (a,b]$, where

$$D_a^{k\alpha} = D_a^{\alpha}.D_a^{\alpha} \ldots D_a^{\alpha} \quad (k\text{-times}).$$

3. TAYLOR COLLOCATION METHOD IN THE FRACTIONAL SENSE

We will now generalize the method in [25,29,30] in order to solve the fractional differential equations. Let us first consider the equation (1) as

$$\sum_{i=0}^{2} P_i(x)\,{}^{C}D_{a+}^{i\alpha} y(x) = f(x) \qquad (a \le x \le b) \tag{11}$$

with initial conditions

$$y^{(k)}(a) = \lambda_k \in R, \quad k = 0,1 \tag{12}$$

where $y(x)$ is the unknown function, the known functions $P_i(x)$ and $f(x)$ are defined on the domain which we are interested and $D^n = d^n / dx^n$ is ordinary differentiation such that $n = \lceil 2\alpha \rceil$ is the value of 2α to be rounded up to the nearest integer.

Suppose that the solution of above problem (11) is

$$y(x) = \sum_{i=0}^{N} \frac{1}{\Gamma(i\alpha+1)} ({}^{C}D_{a+}^{i\alpha} y)(\tau)(x-\tau)^{i\alpha} + R_N^{\alpha}(x) \tag{13}$$

where N is chosen any positive integer with $N \ge 2$ and

$$R_N^{\alpha}(x) = \frac{({}^{C}D_{a+}^{(N+1)} y)(\xi)}{\Gamma((N+1)\alpha+1)}(x-\tau)^{(N+1)\alpha} \tag{14}$$

for $a \le \xi \le x$, $\forall x \in (a,b]$ and ${}^{C}D_{a+}^{i\alpha} = {}^{C}D_{a+}^{\alpha} \cdot {}^{C}D_{a+}^{\alpha} \cdots {}^{C}D_{a+}^{\alpha}$ (i -times).

4. THE MATRIX REPRESENTATIONS

4.1 For the function $y(x)$ and its Caputo Fractional Derivative $({}^{C}D_{a+}^{\alpha} y)(x)$

Let us we have the solution (13) of the equation (11) that can be written in the matrix form

$$[y(x)] = \mathbf{X}\mathbf{M_0}\mathbf{A} \tag{15}$$

where

$$\mathbf{X}(x) = [1 \quad (x-\tau)^{\alpha} \quad (x-\tau)^{2\alpha} \quad \cdots \quad (x-\tau)^{N\alpha}]$$

$$\mathbf{A} = [({}^{C}D_{a+}^{0} y)(\tau) \quad ({}^{C}D_{a+}^{\alpha} y)(\tau) \quad ({}^{C}D_{a+}^{2\alpha} y)(\tau) \quad \cdots \quad ({}^{C}D_{a+}^{N\alpha} y)(\tau)]^{\mathrm{T}}$$

and

$$\mathbf{M}_0 = \begin{bmatrix} \frac{1}{\Gamma(1)} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{\Gamma(\alpha+1)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{\Gamma(2\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\Gamma(N\alpha+1)} \end{bmatrix} \tag{16}$$

To obtain a solution (15), we propose the Taylor Collocation method in the fractional sense as follows. In this method, it is computed the generalized Taylor coefficients by using collocation points and it is found the matrix $\mathbf{A}$ containing the unknown generalized Taylor coefficients.

Now, let us define the collocation points as

$$x_i = a + i\frac{b-a}{N}; \quad i = 0, 1, 2, ..., N, \tag{17}$$

so that $a \le x_i \le b$. Then we substitute the collocation points (17) into (11) to obtain the system

$$\sum_{k=0}^{2} P_k(x_i)({}^{C}D_{a+}^{k\alpha}y)(x_i) = f(x_i)\ ; \quad i = 0, 1, 2, ..., N \tag{18}$$

that can be written in the matrix form

$$\sum_{k=0}^{2} \mathbf{P}_k \mathbf{Y}^{(k\alpha)} = \mathbf{F} \tag{19}$$

where

$$\mathbf{P}_k = \begin{bmatrix} P_k(x_0) & 0 & \cdots & 0 \\ 0 & P_k(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_k(x_N) \end{bmatrix},\ \mathbf{F} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix},\ \mathbf{Y}^{(k\alpha)} = \begin{bmatrix} ({}^{C}D_{a+}^{k\alpha}y)(x_0) \\ ({}^{C}D_{a+}^{k\alpha}y)(x_1) \\ \vdots \\ ({}^{C}D_{a+}^{k\alpha}y)(x_N) \end{bmatrix}.$$

Let us assume that the $k\alpha$ th derivative of the function in (13) with respect to x has the truncated Taylor series expansion

$$({}^{C}D_{a+}^{k\alpha}y)(x_i) = \sum_{n=k}^{N} \frac{1}{\Gamma(k\alpha+n)}({}^{C}D_{a+}^{n\alpha}y)(\tau)(x_i-\tau)^{(n-k)\alpha}; \quad a \le x \le b \tag{20}$$

where $({}^{C}D_{a+}^{k\alpha}y)(\tau)$, $k = 0,1,...,N$ are the generalized Taylor coefficients, and $({}^{C}D_{a+}^{0}y)(\tau) = y(\tau)$. Then substituting the Taylor collocation points into (20), we get the matrix forms

$$[({}^{C}D_{a+}^{k\alpha}y)(x_i)] = \mathbf{X}(x_i)\mathbf{M}_k\mathbf{A}\ , \ (k = 0,1,...,N) \tag{21}$$

or

$$\mathbf{Y}^{(k\alpha)} = \bar{\mathbf{X}}\mathbf{M}_k\mathbf{A} \tag{22}$$

where

$$\bar{\mathbf{X}} = \begin{bmatrix} \mathbf{X}(x_0) \\ \mathbf{X}(x_1) \\ \vdots \\ \mathbf{X}(x_N) \end{bmatrix} = \begin{bmatrix} 1 & (x_0-\tau)^{\alpha} & \cdots & (x_0-\tau)^{N\alpha} \\ 1 & (x_1-\tau)^{\alpha} & \cdots & (x_1-\tau)^{N\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & (x_N-\tau)^{\alpha} & \cdots & (x_N-\tau)^{N\alpha} \end{bmatrix},$$

$$\mathbf{M}_k = \begin{bmatrix} 0 & 0 & \cdots & \overbrace{\frac{1}{\Gamma(1)}}^{k+1 \text{ column}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{\Gamma(\alpha+1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & \frac{1}{\Gamma((N-k)\alpha+1)} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}$$

which are $(N+1)\times(N+1)$ matrices for $k > 0$.

4.2 For the conditions
In view of (22), by substituting

$$[({}^{C}D_{a+}^{k\alpha}y)(a)] = \mathbf{X}(a)\mathbf{M}_k\mathbf{A}$$

into (12), the conditions can also be written in the matrix form as

$$\mathbf{X}(a)\mathbf{M}_k\mathbf{A} = \lambda_k, \qquad k = 0,1,...,\lceil\alpha\rceil$$

and taking

$$\mathbf{X}(a)\mathbf{M}_k = \mathbf{C}_k \equiv \begin{bmatrix} c_{k0} & c_{k1} & \cdots & c_{kN} \end{bmatrix},$$

it can be written

$$\mathbf{C}_k\mathbf{A} = \begin{bmatrix}\lambda_k\end{bmatrix} \tag{23}$$

or the augmented matrices of them are

$$\begin{bmatrix}\mathbf{C}_k;\lambda_k\end{bmatrix} = \begin{bmatrix} c_{k0} & c_{k1} & \cdots & c_{kN} & ; & \lambda_k \end{bmatrix}. \tag{24}$$

5. THE PROCESS OF THE METHOD BY USING THE MATRIX REPRESENTATIONS

By considering (22), we have the matrix equation

$$\left(\sum_{k=0}^{2}\mathbf{P}_k\bar{\mathbf{X}}\mathbf{M}_k\right)\mathbf{A} = \mathbf{F}, \tag{25}$$

and we can also write (25) in the form

$$\mathbf{WA} = \mathbf{F} \text{ or } [\mathbf{W};\mathbf{F}] \tag{26}$$

that corresponds to a system of $(N+1)$ algebraic equations with the unknown generalized Taylor coefficients where

$$\mathbf{W} = [w_{pq}] = \sum_{k=0}^{2}\mathbf{P}_k\bar{\mathbf{X}}\mathbf{M}_k, \; p,q = 0,1,...,N. \tag{27}$$

To obtain the solution of (11) subject to (12), now we have the new augmented matrix by replacing the row matrix (23) by the last row of matrix (26)

$$\begin{bmatrix}\tilde{\mathbf{W}};\tilde{\mathbf{F}}\end{bmatrix} = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & f(x_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & f(x_1) \\ \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N} & ; & f(x_{N-m}) \\ c_{00} & c_{01} & \cdots & c_{00} & ; & \lambda_0 \\ \vdots & \vdots & \ddots & \vdots & ; & \vdots \\ c_{m-1,0} & c_{m-1,1} & \cdots & c_{m-1,N} & ; & \lambda_{m-1} \end{bmatrix}. \tag{28}$$

If $rank\,\tilde{\mathbf{W}} = rank[\tilde{\mathbf{W}};\tilde{\mathbf{F}}] = N+1$ in (28), then we can write

$$\mathbf{A} = (\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{F}} \tag{29}$$

where it can be uniquely determined. If $\det(\tilde{\mathbf{W}}) = 0$, then there is no solution and the method cannot be used or we may obtain the particular solutions by means of the system.

6. THE ERROR OF THE GENERALIZED TAYLOR POLYNOMIAL

The accuracy of the obtained solutions can be checked by using (14) which is increased when the large N is chosen and is decreased as the value of x moves away from the center τ [6]. The obtained polynomial expansion is an approximate solution when the function $y(x)$ and its derivative $y^{(k\alpha)}(x)$ are substituted in Eq. (1), the resulting equation must be satisfied approximately: that is, for the collocation points $x = x_i \in [a,b]$ $i = 0,1,...,N$.

$$E(x_i) = \left| \sum_{k=0}^{2} P_k(x_i)({}^C D_{a+}^{k\alpha} y)(x_i) - f(x_i) \right| \cong 0$$

or

$$E(x_i) \le 10^{k_i} \qquad (k_i \text{ is any positive integer})$$

If $\max(10^{k_i}) = 10^{-k}$, (k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $E(x_i)$ at each of points x_i becomes smaller than the prescribed 10^{-k} [25,29,30].

7. NUMERICAL EXAMPLE

Example Firstly we consider the problem in [26] as the functions $P_0(x) = 1,\ P_1(x) = 1,\ P_2(x) = 0$ and $\alpha = 0.5$, $f(x) = x^2 + \frac{2}{\Gamma(2.5)} x^{1.5}$ are taken in (1):

$$D^{0.5} y(x) = -y(x) + x^2 + \frac{2}{\Gamma(2.5)} x^{1.5}$$

with the condition $y(0) = 0$. In order to solve the problem by using the proposed method in Section 3 and 4, considering the collocation points for $N = 6$, we firstly define the matrix representations of $y(x)$, ${}^C D_{0+}^{0.5} y(x)$, $f(x)$ and initial condition that are written from (19) where

$$\mathbf{F} = \begin{bmatrix} 0 & \frac{2}{27}\frac{\sqrt{6}}{\sqrt{\pi}} + \frac{1}{36} & \frac{8}{27}\frac{\sqrt{3}}{\sqrt{\pi}} + \frac{1}{9} & \frac{2}{3}\frac{\sqrt{2}}{\sqrt{\pi}} + \frac{1}{4} & \frac{16}{27}\frac{\sqrt{6}}{\sqrt{\pi}} + \frac{4}{9} & \frac{10}{27}\frac{\sqrt{30}}{\sqrt{\pi}} + \frac{25}{36} & \frac{8}{3\sqrt{\pi}} + 1 \end{bmatrix}^{\mathbf{T}}$$

$$\mathbf{P}_0 = \mathbf{P}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \overline{\mathbf{X}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \left(\frac{1}{6}\right)^{\alpha} & \left(\frac{1}{6}\right)^{2\alpha} & \left(\frac{1}{6}\right)^{3\alpha} & \left(\frac{1}{6}\right)^{4\alpha} & \left(\frac{1}{6}\right)^{5\alpha} & \left(\frac{1}{6}\right)^{6\alpha} \\ 1 & \left(\frac{1}{3}\right)^{\alpha} & \left(\frac{1}{3}\right)^{2\alpha} & \left(\frac{1}{3}\right)^{3\alpha} & \left(\frac{1}{3}\right)^{4\alpha} & \left(\frac{1}{3}\right)^{5\alpha} & \left(\frac{1}{3}\right)^{6\alpha} \\ 1 & \left(\frac{1}{2}\right)^{\alpha} & \left(\frac{1}{2}\right)^{2\alpha} & \left(\frac{1}{2}\right)^{3\alpha} & \left(\frac{1}{2}\right)^{4\alpha} & \left(\frac{1}{2}\right)^{5\alpha} & \left(\frac{1}{2}\right)^{6\alpha} \\ 1 & \left(\frac{2}{3}\right)^{\alpha} & \left(\frac{2}{3}\right)^{2\alpha} & \left(\frac{2}{3}\right)^{3\alpha} & \left(\frac{2}{3}\right)^{4\alpha} & \left(\frac{2}{3}\right)^{5\alpha} & \left(\frac{2}{3}\right)^{6\alpha} \\ 1 & \left(\frac{5}{6}\right)^{\alpha} & \left(\frac{5}{6}\right)^{2\alpha} & \left(\frac{5}{6}\right)^{3\alpha} & \left(\frac{5}{6}\right)^{4\alpha} & \left(\frac{5}{6}\right)^{5\alpha} & \left(\frac{5}{6}\right)^{6\alpha} \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\mathbf{M}_0 = \begin{bmatrix} \frac{1}{\Gamma(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\Gamma(1+\alpha)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\Gamma(1+2\alpha)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\Gamma(1+3\alpha)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\Gamma(1+4\alpha)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\Gamma(1+5\alpha)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\Gamma(1+6\alpha)} \end{bmatrix},$$

$$\mathbf{M}_1=\begin{bmatrix}0 & \frac{1}{\Gamma(1)} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{\Gamma(1+\alpha)} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{\Gamma(1+2\alpha)} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{\Gamma(1+3\alpha)} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\Gamma(1+4\alpha)} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\Gamma(1+5\alpha)}\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

and for the condition, the augmented matrix form is

$$[\mathbf{C}_0;\lambda_0]=[1\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad ;\quad 0].$$

The approximate solution of $y(x)$ by the generalized Taylor polynomial in fractional sense with truncation error is

$$y(x)=\sum_{i=0}^{6}\frac{1}{\Gamma(i\alpha+1)}({}^{C}D_{a+}^{i\alpha}y)(\tau)(x-\tau)^{i\alpha}$$

where $\alpha=\frac{1}{2}$ and $\tau=0$. The matrix form of the problem is defined by

$$\left\{\mathbf{P}_0\bar{\mathbf{X}}\mathbf{M}_0+\mathbf{P}_1\bar{\mathbf{X}}\mathbf{M}_1\right\}\mathbf{A}=\mathbf{F}.$$

After the system of the augmented matrices and the condition are computed, we obtain the new augmented matrix in the form

$$\left[\tilde{\mathbf{W}};\tilde{\mathbf{F}}\right]=\begin{bmatrix}1 & 1 & 0 & 0 & 0 & 0 & 0 & ; & 0\\ 1 & \frac{1}{3}\frac{\sqrt{6}}{\sqrt{\pi}}+1 & \frac{1}{3}\frac{\sqrt{6}}{\sqrt{\pi}}+\frac{1}{6} & \frac{1}{27}\frac{\sqrt{6}}{\sqrt{\pi}}+\frac{1}{6} & \frac{1}{27}\frac{\sqrt{6}}{\sqrt{\pi}}+\frac{1}{72} & \frac{1}{405}\frac{\sqrt{6}}{\sqrt{\pi}}+\frac{1}{72} & \frac{1}{405}\frac{\sqrt{6}}{\sqrt{\pi}}+\frac{1}{1296} & ; & \frac{2}{27}\frac{\sqrt{6}}{\sqrt{\pi}}+\frac{1}{36}\\ 1 & \frac{2}{3}\frac{\sqrt{3}}{\sqrt{\pi}}+1 & \frac{2}{3}\frac{\sqrt{3}}{\sqrt{\pi}}+\frac{1}{3} & \frac{4}{27}\frac{\sqrt{3}}{\sqrt{\pi}}+\frac{1}{3} & \frac{4}{27}\frac{\sqrt{3}}{\sqrt{\pi}}+\frac{1}{18} & \frac{8}{405}\frac{\sqrt{3}}{\sqrt{\pi}}+\frac{1}{18} & \frac{8}{405}\frac{\sqrt{3}}{\sqrt{\pi}}+\frac{1}{162} & ; & \frac{8}{27}\frac{\sqrt{3}}{\sqrt{\pi}}+\frac{1}{9}\\ 1 & \frac{\sqrt{2}}{\sqrt{\pi}}+1 & \frac{\sqrt{2}}{\sqrt{\pi}}+\frac{1}{2} & \frac{1}{3}\frac{\sqrt{2}}{\sqrt{\pi}}+\frac{1}{2} & \frac{1}{15}\frac{\sqrt{2}}{\sqrt{\pi}}+\frac{1}{8} & \frac{1}{15}\frac{\sqrt{2}}{\sqrt{\pi}}+\frac{1}{8} & \frac{1}{15}\frac{\sqrt{2}}{\sqrt{\pi}}+\frac{1}{48} & ; & \frac{2}{3}\frac{\sqrt{2}}{\sqrt{\pi}}+\frac{1}{4}\\ 1 & \frac{2}{3}\frac{\sqrt{6}}{\sqrt{\pi}}+1 & \frac{2}{3}\frac{\sqrt{6}}{\sqrt{\pi}}+\frac{2}{3} & \frac{8}{27}\frac{\sqrt{6}}{\sqrt{\pi}}+\frac{2}{3} & \frac{32}{405}\frac{\sqrt{6}}{\sqrt{\pi}}+\frac{2}{9} & \frac{32}{405}\frac{\sqrt{6}}{\sqrt{\pi}}+\frac{2}{9} & \frac{32}{405}\frac{\sqrt{6}}{\sqrt{\pi}}+\frac{4}{81} & ; & \frac{16}{27}\frac{\sqrt{6}}{\sqrt{\pi}}+\frac{4}{9}\\ 1 & \frac{1}{3}\frac{\sqrt{30}}{\sqrt{\pi}}+1 & \frac{1}{3}\frac{\sqrt{30}}{\sqrt{\pi}}+\frac{5}{6} & \frac{5}{27}\frac{\sqrt{30}}{\sqrt{\pi}}+\frac{5}{6} & \frac{5}{81}\frac{\sqrt{30}}{\sqrt{\pi}}+\frac{25}{72} & \frac{5}{81}\frac{\sqrt{30}}{\sqrt{\pi}}+\frac{25}{72} & \frac{5}{81}\frac{\sqrt{6}}{\sqrt{\pi}}+\frac{125}{1296} & ; & \frac{10}{27}\frac{\sqrt{30}}{\sqrt{\pi}}+\frac{25}{36}\\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & ; & 0\end{bmatrix}.$$

Once proceeds the procedure in Section 4, this system has the solution

$$\mathbf{A}=[0\quad 0\quad 0\quad 0\quad 2\quad 0\quad 0]^{\mathbf{T}}.$$

Therefore, from (29) we find the exact solution

$$y(x)=x^2.$$

8. CONCLUSION

In this paper, we obtain the matrix formulation of the generalized Taylor collocation method for fractional differential equation (1) of order 2α. For this reason, we use Taylor collocation method in the fractional sense and we consider Caputo fractional derivative as $({}^{C}D_{a+}^{\alpha}y)(x)$ for the fractional derivative. After obtaining the matrix representations it is concluded that the coefficients of the generalized Taylor method can be found by using the resulting equation (29). With the specific example it is seen that the solution is the exact one.

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