ASYMTOTIC NORMALITY OF ESTIMATORS IN HETEROSCEDASTIC ERRORS-IN-VARIABLES MODEL FOR NA SAMPLES

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ABSTRACT
This article is concerned with the estimating problem of heteroscedastic partially linear errors-in-variables models. We derive the asymptotic normality for estimators of the slope parameter and the nonparametric component in the case of known error variance with NA(negatively associated) random errors. Also, when the error variance is unknown, the asymptotic normality for the estimators of the slope parameter and the nonparametric component as well as variance function is considered under independent assumptions. Finite sample behavior of the estimators is investigated via simulations too.

Keywords: *Partially linear errors-in-variables model, Negatively associated, Asymptotic normality, Heteroscedastic, Least-squares estimator.*
MSC: 62J12 · 62E20

1. INTRODUCTION

Consider the following heteroscedastic partially linear errors-in-variables (EV) model

$$\begin{cases} y_i = & \xi_i\beta + g(t_i) + \varepsilon_i, \\ x_i = & \xi_i + \mu_i. \end{cases} \tag{1}$$

where $\varepsilon_i = \sigma_i e_i, \sigma_i^2 = f(u_i)$, (ξ_i, t_i, u_i) are nonrandom design points, (t_i, x_i, y_i) are observed samples, β is an unknown parameter to be estimated, $\{\xi_i\}$ are the potential variables cannot be observed, $\{y_i\}$ are the response variables, $\{x_i\}$ are observed with measurement errors $\{\mu_i\}$ and with $E\mu_i = 0$, and $\{e_i\}$ are random errors and with $Ee_i = 0$. Assume that there is a function $h(\cdot)$ defined on closed interval $[0,1]$ satisfying

$$\xi_i = h(t_i) + v_i. \tag{2}$$

where $\{v_i\}$ are also unknown design points.

Model (1) and its special cases have been widely studied by many authors. When the $\{\xi_i\}$ can be observed, $\sigma_i^2 = \sigma^2$, and the errors $\{e_i\}$ are independent identically distribution(i.i.d), the model reduces to the homoscedastic partially linear regression model, which was studied by Engle et al (1986)[1]. When $g(t) \equiv 0$, $\sigma_i^2 = f(u_i)$, the model becomes into heteroscedastic partially linear regression model, which was extensively studied by Carroll (1982)[2], Robinson (1987)[3]. In addition, when $g(t) \neq 0$, and the $\{\xi_i\}$ can not de directly observed, the model (1) degenerates into partially linear EV model, which can be seen in Cui and Li (1998)[4], Wang (1999)[5], Liang (1999)[6] and so on.

In recent decades, semi-parametric EV models have been widely concenred. Miao, Zhang and Wang(2013)[7] considered the strong consistency and asymptotic normality for the least square estimators in a linear EV regression model; Liu and Chen(2005)[8] discussed the consistency of estimators and derived the equivalence relation of weak or strong consistency for the estimators; Cui(2006)[9] summarized the T regression estimate and EM arithmetic in a linear EV regression model; Many of early results of the study of EV model can be seen in Fuller (1987)[10], Cheng and Van Ness (1999)[11] and Carrol (1995)[12].

In this paper, we consider the estimation problem for model (1) under the errors $\{e_i, 1\}$ being mean zero negatively associated(NA) random variables. A finite family of random variables $\{X_i, 1\}$ is said to be NA random variables if for every pair of disjoint subsets A and B of {1,2,...,n}, we have

$$Cov(f_1(X_i, i \in A), f_2(X_j, j \in B))0$$

whenever f_1 and f_2 are coordinatewise increasing function and such that the covariance exists. An infinite family of random variables is NA if every finite subfamily is NA.

The NA view was introduced by Alam and Saxena (1981)[13], and Joag-Dev and Proschan (1983)[14] discovered the the character of multivariate distribution of NA sequence and discovered fundamental properties; Liang (2000)[15] discovered complete convergence; NA sequence not only has been applied in the multivariat statistical analysis, but also in the oceans, weather, and other engineering fields, risk analysis and time series analysis just as the same as other positive and negative dependent sequence. However, there are few asymptotic results for the estimators of parametric and nonparametric components in partial linear EV model regressions under NA error's structure.

The paper is organized as follows. In Section 2, we list some assumptions. The main results are given in

Section 3. A simulation study is presented in section 4. Some preliminary lemmas are stated in Section 5. Proofs of the main results are provided in Sections 6.

2. ASSUMPTIONS

First, we assume that $\{t_i, h_i, v_i, g_i, \varepsilon_i, \mu_i, \xi_i, 1\}$ satisfy model (1), and that $W_{ni}(\cdot)(1)$ are some weight functions defined on I and set $\tilde{h}_i = h(t_i) - \sum_{j=1}^{n} W_{nj}(t_i)h(t_j)$, $\tilde{v}_i = v_i - \sum_{j=1}^{n} W_{nj}(t_i)v_j$, $\tilde{g}_i = g(t_i) - \sum_{j=1}^{n} W_{nj}(t_i)g(t_j)$, $\tilde{\varepsilon}_i = \varepsilon_i - \sum_{j=1}^{n} W_{nj}(t_i)\varepsilon_j$, $\tilde{\mu}_i = \mu_i - \sum_{j=1}^{n} W_{nj}(t_i)\mu_j$ and $\tilde{\xi}_i = \xi_i - \sum_{j=1}^{n} W_{nj}(t_i)\xi_j$. Then, we shall list some conditions, which will be used in the paper.

•

- Let $\{e_i, 1\}$ be a sequence of NA random variables with mean zero, and let $\{\mu_i, 1\}$ be a sequence of independent random variables with mean zero, and $\{e_i, 1\}$ is independent with $\{\mu_i, 1\}$. Assume that $Ee_i^2 = 1$, $\sup_i E|e_i|^p < \infty$, for some $p > 4$, $\sup_i E|\mu_i|^p < \infty$, for some $p > 4$, and the $E\mu_i^2 = \Xi_\mu^2 > 0$ is known.

- Let both of $\{e_i, 1\}$ and $\{\mu_i, 1\}$ be sequences of independent random variables with mean zero, $Ee_i^2 = 1$, $E\mu_i^2 = \Xi_\mu^2 > 0$ and $\sup_i Ee_i^6 + \sup_i E\mu_i^6 < \infty$. $\{\mu_i, 1\}$ is independent of $\{e_j, 1\}$.

• Let $\{v_i, 1\}$ in condition (1.2) be a sequence satisfying
- $\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} v_i^2 = \Sigma_0 (0 < \Sigma_0 < \infty)$;
- $\lim_{n\to\infty} \sup_n (\sqrt{n}\log n)^{-1} \cdot \max_1 |\sum_{i=1}^{m} v_{j_i}| < \infty$.

•
- $0 < m_0 \min_1 f(u_i) \max_1 f(u_i)_0 < \infty$;
- $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ are continuous function and satisfy the first-order Lipschitz condition on I.

• The probability weight functions $W_{nj}(t_i)$ be weight functions defined on $[0,1]$ and satisfy
- $\max_1 \sum_{i=1}^{n} W_{nj}(t_i) = O(1)$;
- $\max_1 \sum_{j=1}^{n} W_{nj}(t_i) I(|t_i - t_j| > n^{-1/4}) = o(n^{-1/4})$;
- $\max_{1,j} W_{nj}(t_i) = o(n^{-1/2}\log^{-1} n)$,
- $\max_{1,j} W_{nj}(t_i) = O(n^{-s})$.

• Let $\widehat{W}_{ni}(\cdot)(1)$ be weight functions defined on I. Conditons A3(i)(ii)(iv) are satisfied replacing t_i and W_{ni} by u_i and $\widehat{W}_{ni}$, respectively.

Remark 2.1 *Conditions (A0)-(A3) are standard regularity conditions and used commonly in the literature, see Gao et al.(1994)[16] and Chen et al.(1998)[17];*

Remark 2.2 *Under some mild conditions, the following two weight functions satisfy hypothesis (A3):*

$$W_{ni}^{(1)}(t) = \frac{1}{h}\int_{s_i-1}^{s_i} K\left(\frac{t-s}{h_n}\right)ds,$$

$$W_{ni}^{(2)}(t) = K\left(\frac{t-t_i}{h_n}\right)\left[\sum_{j=1}^{n} K\left(\frac{t-t_i}{h_n}\right)\right]^{-1}.$$

where $s_i = (t_i + t_{i-1})/2, i = 1,2,\ldots,n-1, s_0 = 0, s_n = 1, K(\cdot)$ is the Parzen-Rosenblatt kernel function, we can see Parzen(1962)[18], and the h_n is a bandwidth parameter.

3. MAIN RESULTS

For model (1), we want to seek the estimator of β and $g(\cdot)$. Firstly, when the error are homoscedastic and the ξ_i can be observed, we can apply the least squares estimation method to estimate the parameter β. On the hand, we assume the parameter β is known, and then to estimate $g(\cdot)$; for each given β, we have $g(t_i) = E(y_i - x_i\beta)$,1. Therefore, based on the (x_i, t_i, y_i), we can define the estimator of $g(\cdot)$, that is $g_n^*(t,\beta) = \sum_{i=1}^n W_{ni}(t)(y_i - x_i\beta)$. Then, based on the model (1), we can also define the LSE of β by following formula:

$$\sum_{i=1}^n [y_i - x_i\beta - g_n^*(t_i,\beta)]^2 - \Xi_\mu^2\beta^2 = min!$$

On the other hand, under this condition of partially linear EV model, Liang et al.(1999)[?] improved the LSE on the basis of the usually partially linear model, and employ the estimator of parameter β, write that

$$\hat{\beta}_L = [\sum_{i=1}^n (\tilde{x}_i^2 - \Xi_\mu^2)]^{-1} \sum_{i=1}^n \tilde{x}_i\tilde{y}_i. \tag{1}$$

where $\tilde{x}_i = x_i - \sum_{j=1}^n W_{nj}(t_i)x_j$, $\tilde{y}_i = y_i - \sum_{j=1}^n W_{nj}(t_i)y_j$.

Secondly, when the errors are heteroscedastic, we consider two different cases according to $f(\cdot)$. If $\sigma_i^2 = f(u_i)$ are known, then the $\hat{\beta}_L$ is medified to be the weighted least-squares estimator (WLSE)

$$\hat{\beta}_{W_1} = [\sum_{i=1}^n \sigma_i^{-2}(\tilde{x}_i^2 - \Xi_\mu^2)]^{-1} \sum_{i=1}^n \sigma_i^{-2}\tilde{x}_i\tilde{y}_i. \tag{2}$$

In fact, the $\sigma_i^2 = f(u_i)$ are unknown and must be estimated. In the case, suppose that $Ee_i^2 = 1$, we have $E[y_i - \xi_i\beta - g(t_i)]^2 = f(u_i)$. Therefore, the estimator of $f(u_i)$ can be defined by

$$\hat{f}_n(u_i) = \sum_{j=1}^n \widehat{W}_{nj}(u_i)(\tilde{y}_j - \tilde{x}_j\hat{\beta}_L)^2 - \Xi_\mu^2\hat{\beta}_L^2. \tag{3}$$

For convenience, we assume that $\min_1 \hat{f}_n(u_i) > 0$. Then we can define a nonparametric estimator of σ_i^2, $\hat{\sigma}_{ni}^2 = \hat{f}_n(u_i)$.

In consequence, when the errors are heteroscedastic and unknown, the WLSE of β is

$$\hat{\beta}_{W_2} = [\sum_{i=1}^n \hat{\sigma}_{ni}^{-2}(\tilde{x}_i^2 - \Xi_\mu^2)]^{-1} \sum_{i=1}^n \hat{\sigma}_{ni}^{-2}\tilde{x}_i\tilde{y}_i. \tag{4}$$

Meanwhile, using $\hat{\beta}_L$, $\hat{\beta}_{W_1}$, $\hat{\beta}_{W_2}$, we can define the three estimators for $g(\cdot)$:

$$\hat{g}_L(t) = \sum_{i=1}^n W_{ni}(t)(y_i - x_i\hat{\beta}_L), \tag{5}$$

$$\hat{g}_{W_1}(t) = \sum_{i=1}^n W_{ni}(t)(y_i - x_i\hat{\beta}_{W_1}), \tag{6}$$

$$\hat{g}_{W_2}(t) = \sum_{i=1}^n W_{ni}(t)(y_i - x_i\hat{\beta}_{W_2}). \tag{7}$$

In this paper, we provide some notions and a definition that will be used in the process of proof.

$$\eta_i = \varepsilon_i - \mu_i\beta, S_n^2 = \sum_{i=1}^n \tilde{\xi}_i^2, T_n^2 = \sum_{i=1}^n \sigma_i^{-2}\tilde{\xi}_i^2,$$
$$S_{1n}^2 = \sum_{i=1}^n (\tilde{x}_i^2 - \Xi_\mu^2), \Sigma_{1n}^2 = \mathrm{V}ar[\sum_{i=1}^n \sigma_i^{-2}(\tilde{\xi}_i + \mu_i)(\varepsilon_i - \mu_i\beta)],$$
$$\Gamma_n^2(t) = \mathrm{V}ar[\sum_{i=1}^n W_{ni}(t)(\varepsilon_i - \mu_i\beta)], \Delta_n^2(u) = \sum_{i=1}^n \widehat{W}_{ni}^2(u)\mathrm{V}ar[(\varepsilon_i - \mu_i\beta)^2]. \tag{8}$$

Definition 3.1 *Let* $\{X_t, t = 0, \pm1, \pm2, \cdots\}$ *be a strictly stationary time series. For* $n = 1,2,\cdots,$ *define*

$$\rho(n) = \sup_{X\in L^2(F_{-\infty}^0), Y\in L^2(F_n^\infty)} |Corr(X,Y)|$$

where F_i^j denotes the σ-algebra generated by $\{X_t, i\}$, and $L^2(F_i^j)$ consists of F_i^j-measurable random variables with finite second moment.

When $f(\cdot)$ is known, we give tne asymptotic normality for least-sqares estimators and weighted least-squares estimators of β and $g(\cdot)$.

Theorem 3.1 *Suppose that (A0)(i), (A1), (A2) and (A3) are satisfied. Then we have*

- If Σ_n^2, then $S_n^2(\hat{\beta}_L - \beta)/\Sigma_n \xrightarrow{D} N(0,1)$;
- If Σ_{1n}^2, then $T_n^2(\hat{\beta}_{W_1} - \beta)/\Sigma_{1n} \xrightarrow{D} N(0,1)$.

Theorem 3.2 *Suppose that (A0)(i), (A1), (A2) and (A3) are satisfied. If* $n\Gamma_n^2(t) \to \infty$ *and* $\sum_{i=1}^n W_{ni}^2(t) = O(\Gamma_n^2(t))$, *then we have*

- $[\hat{g}_L(t) - E\hat{g}_L(t)]/\Gamma_n(t) \xrightarrow{D} N(0,1)$;
- $[\hat{g}_{W_1}(t) - E\hat{g}_{W_1}(t)]/\Gamma_n(t) \xrightarrow{D} N(0,1)$.

Remark 3.1 *According to Zhang and Liang(2013) Remark (3.1), we think* $\sum_n^2 Cn, \sum_{1n} Cnandn\Gamma_n^2(t) \to \infty$ *is reasonable.*

When $f(\cdot)$ is unknown, we give tne asymptotic normality for the estimators of β, $g(\cdot)$ and $f(\cdot)$ under the $\{e_i, 1\}$ is an independent sequence. And the proof of the Theorem 3.3, 3.4 and 3.5 , we can reference the Zhang and Liang (2013)[19].

Theorem 3.3 *Suppose that (A0)(ii), (A1), (A2) and (A4) and (A3)(i)(ii)(iv) for some* $1/2 < s < 1$ *are satisfied. Then* $T_n^2(\hat{\beta}_{W_2} - \beta)/\Sigma_{1n} \xrightarrow{D} N(0,1)$.

Theorem 3.4 *Suppose that (A0)(ii), (A1), (A2), (A4) and (A3)(i)(ii)(iv) for some* $5/8 < s < 1$ *are satisfied. Assume that* $max_1|v_i| = O(n^{1/3})$. *For each* $t \in [0,1]$, *if* $n\sum_{i=1}^{n} W_{ni}^2(t) \to \infty$, *then we have* $[\hat{g}_{W_2}(t) - E\hat{g}_{W_2}(t)]/\Gamma_n(t) \xrightarrow{D} N(0,1)$.

Theorem 3.5 *Suppose that (A0)(ii), (A1), (A2), (A4) and (A3)(i)(ii)(iv) for some* $s = 1/2$ *are satisfied. Assume that* $sup_i E\mu_i^8 < \infty$ *and* $inf_i Var[(\varepsilon_i - \mu_i\beta)^2] > 0$. *For each* $u \in [0,1]$, *if* $n\sum_{i=1}^{n} \widehat{W}_{ni}^2(u) \to \infty$, *then we have* $[\hat{f}_n(u) - E\hat{f}_n(u)]/\Delta_n(u) \xrightarrow{D} N(0,1)$.

4. SIMULATION STUDY

In this section, we carry out a simulation to study the finite sample performance of the proposed estimators. In particular, We examine how good the asymptotic normality is for the estimators of β, $g(\cdot)$ by Q-Q plot.

Observations are generated from

$$\begin{cases} y_i = & \xi_i\beta + g(t_i) + \varepsilon_i, \\ x_i = & \xi_i + \mu_i, i = 1,2,\cdots,n, \end{cases}$$

where $\beta = 1$, $g(t) = \sin(2\pi t)$, $\sigma_i^2 = f(u_i)$, $f(u) = [1 + 0.5\cos(2\pi u)]^2$, $t_i = (i - 0.5)/n$ and $u_i = (i-1)/n$, $\xi_i = t_i^2 + v_i$ with $v_i = \sin(i)/(n^{1/3})$. $\{\mu_i, 1\}$ is an i.i.d. $N(0, 0.2^2)$ sequence. $\{ e_i, 1 \}$ are subjected to multivariate normal distribution with $E(e_1,\cdots,e_n) = (0,\cdots,0)$, $Cov(e_i,e_j) = -4^{-(j-i)-1} for i \neq j$ and $Var(e_i) = 0.5^2 for 1$. For the proposed estimators, the weight functions are taken as

$$W_{ni}(t) = \frac{K((t-t_i)/h_n)}{\sum_{j=1}^{n} K((t-t_j)/h_n)}, \widehat{W}_{ni}(u) = \frac{K((u-u_i)/b_n)}{\sum_{j=1}^{n} K((u-u_j)/b_n)}.$$

where $K(\cdot)$ is a Gaussian kernel function, h_n and b_n are two bandwidth sequences.

It is well known that an important issue is the selection of an appropriate bandwidth sequences. This issue has been extensively stuied in the context of nonparametric regression. One of bandwidth selection rules is the delete-one cross-validation rule. It is noted that our estimators may involve two bandwidths. Hence, it is somewhat complicated to select appropriate bandwidths for our estimatos. we state the procedure in the following three steps:

- Select h_n by minimizing

$$CV_1(h_n) = \frac{1}{n}\sum_{i=1}^{n} (y_i - x_i\hat{\beta}_{L,-i} - \hat{g}_{L,-i}(t_i))^2$$

where $\hat{\beta}_{L,-i}$ and $\hat{g}_{L,-i}(t_i)$ are "Leave one out" versions of $\hat{\beta}_L$ and $\hat{g}_L(t_i)$.

- Select h'_n by minimizing

$$CV_2(h'_n) = \frac{1}{n}\sum_{i=1}^{n} (y_i - x_i\hat{\beta}_{W1,-i} - \hat{g}_{W1,-i}(t_i))^2$$

where $\hat{\beta}_{W1,-i}$ and $\hat{g}_{W1,-i}(t_i)$ are "Leave one out" versions of $\hat{\beta}_{W1}$ and $\hat{g}_{W1}(t_i)$.

- Select b_n by minimizing

$$CV_3(b_n) = \frac{1}{n}\sum_{i=1}^{n} (y_i - x_i\hat{\beta}_{W2,-i} - \hat{g}_{W2,-i}(t_i))^2$$

where $\hat{\beta}_{W2,-i}$ and $\hat{g}_{W2,-i}(t_i)$ are "Leave one out" versions of $\hat{\beta}_{W2}$ and $\hat{g}_{W2}(t_i)$.

We found by calculation the corresponding optimal bandwidths $h_1 = 0.38$ and $h_2 = 0.14$.

We give the Q-Q plot for the estimator of β and $g(\cdot)$ under the condition that $f(\cdot)$ is known. In Figure 1, we give the Q-Q plot for $\hat{\beta}_L$ and $\hat{\beta}_{W_1}$ with $n = 100,300$ and 500, respectively. In Figure 2, we provide the Q-Q plot for $\hat{g}_L(\cdot)$ and $\hat{g}_{W_1}(\cdot)$ with $n = 100,300$ and 500, respectively.

From Figure 1-2, we can see that:

- The asymptotic normality of $\hat{\beta}_L$ or $\hat{\beta}_{W_1}$ is obvious, so does the asymptotic normality of $\hat{g}(\cdot)$;
- The normality becomes more obvious with increasing sample size n.

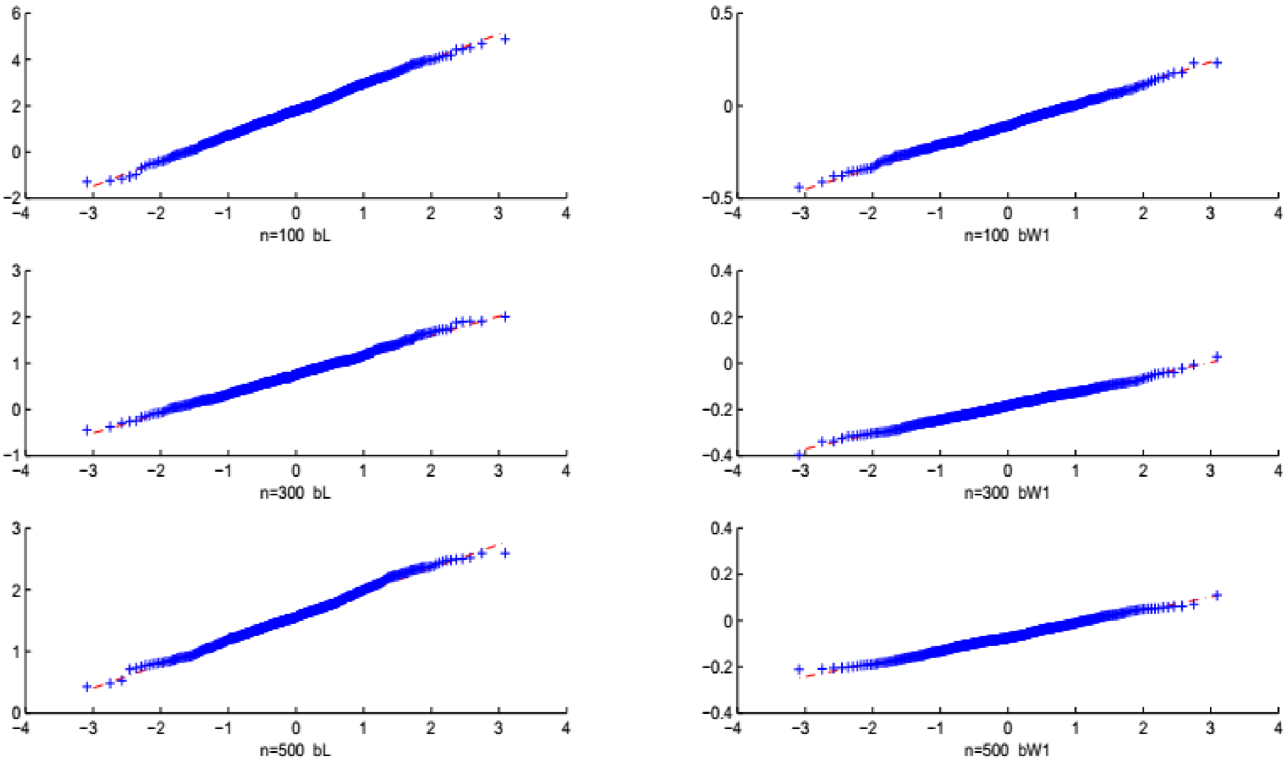

Figure 1: The Q-Q plots for $\hat{\beta}_L$ and $\hat{\beta}_{W_1}$ with N=500, n=100,300 and 500, respectively.

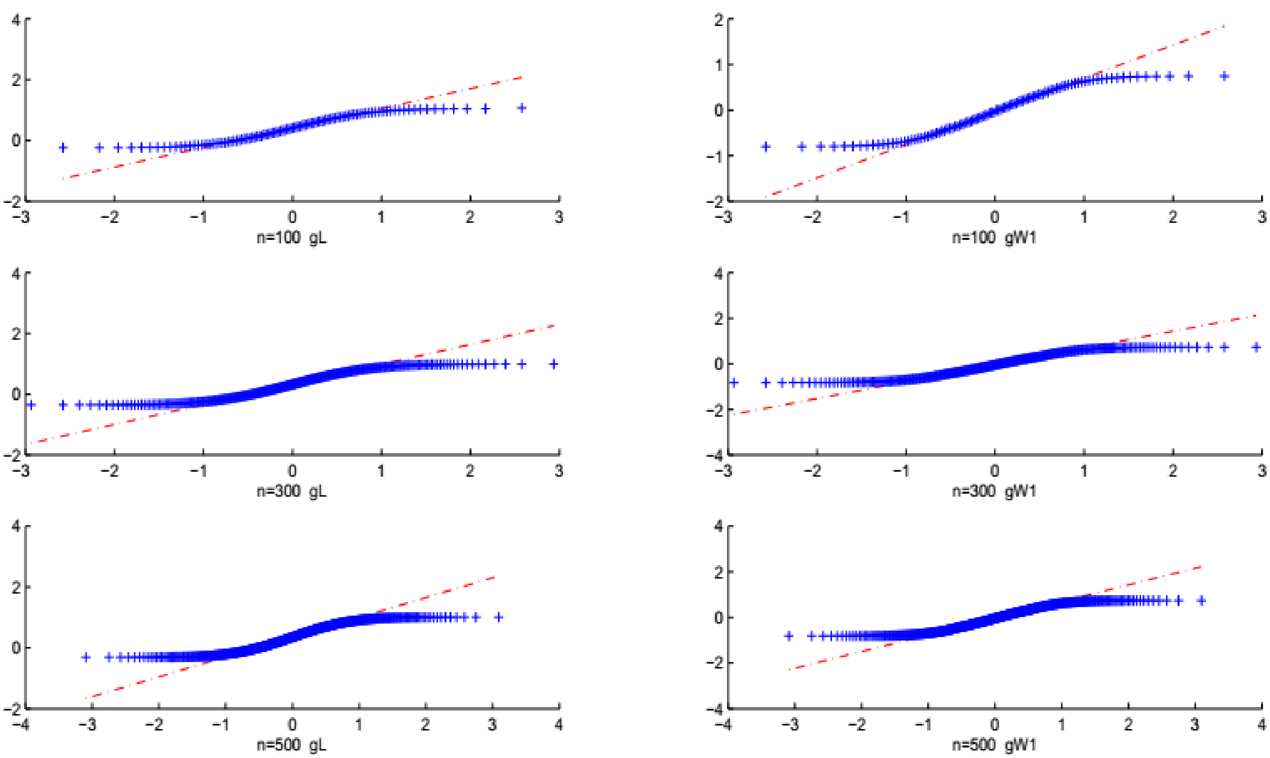

Figure 2: The Q-Q plots for $g_L(\cdot)$ and $g_{W_1}(\cdot)$ with N=500, n=100,300 and 500, respectively.

5. PRELIMINARY LEMMAS

In the sequel, let $c, c_1, \cdots$ and $C, C_1, \cdots$ are some finite positive constants, whose values are unimportant and may change. $a_n = O(b_n)$ means $|a_n||b_n|$, while $a_n = o(b_n)$ means $a_n/b_n \to 0$. $a^+ = \max(0, a)$, $a^- = \max(0, -a)$. And let $\{e_i, 1\}$ be a sequence o f zero mean stationary NA random errors. Now, we introduce several lemmas, which will be used in the proof of the main results.

Lemma 5.1 (Baek and Liang (2006)and Baek(2006), Lemma 3.1) *Let $\alpha > 2$. Assume that $\{a_{ni}, 1, n1\}$ is a triangular array of real numbers with $max_1|a_{ni}| = O(n^{-1/2})$ and $\sum_{i=1}^n a_{ni}^2 = o(n^{-2/\alpha}(logn)^{-1})$. If $sup_i E|e_i|^p < \infty$ for some $p > 2\alpha/(\alpha-1)$. Then*

$$\sum_{i=1}^n a_{ni}e_i = o(n^{-1/\alpha})a.s.$$

Remark 5.1 *In Lemma 5.1, it is quite clear that $p > 2$ as $\alpha \to \infty$ and $\sum_{ni}^n a_{ni}e_i = o(1)$ a.s.; and $p > 4$ when $\alpha > 4$ and $\sum_{ni}^n a_{ni}e_i = o(n^{-1/4})$ a.s. In addition, if all of the "o" is changed into "O", then the conclusion is also right.*

Lemma 5.2 (Hardle et al. (2000)(2000), Lemma A.3) *Let $V_1, \cdots, V_n$ be independent random variables with $EV_i = 0$, finite variances and $sup_1 E|V_j|^r < \infty (r > 2)$. Assume that $\{a_{ki}, k, i = 1, \cdots, n\}$ is a sequence of real numbers such that $sup_{1,k}|a_{ki}| = O(n^{-p_1})$ for some $0 < p_1 < 1$ and $\sum_{j=1}^n a_{ji} = O(n^{p_2})$ for $p_2 max(0, 2/r - p_1)$. Then*

$$\max_1 |\sum_{k=1}^n a_{ki}V_k| = O(n^{-s}\log n)a.s. for\ s = (p_1 - p_2)/2.$$

Lemma 5.3 (Liu and Gan(2003)and Gan(2006)) *Assume a_n is a array of positive real numbers, and $\sum_{n=1}^\infty \sigma_n^2/a_n^2 < \infty$, where $\sigma_n^2 = Var(e_n)$. If $0 < a_n \uparrow \infty$. Then*

$$\sum_{i=1}^n \frac{e_i}{a_n} = o(1)a.s.$$

Lemma 5.4 (Han-Ying Liang, Volker Mammitzsch and Josef Steinebach (2006)and Volker(2006), Lemma 4.4(ii)) *Let $\{a_{ni}, 1, n1\}$ be an array of real numbers and set $\Delta_n^2 = Var(\sum_{i=1}^n a_{ni}e_i)$. Assume that $\sum_{j:|k-j|} |cov(e_k, e_j)| \to 0 asu \to \infty$ uniformly for $k1$, and $max_1|a_{ni}| = o(\Delta_n)$, $\sum_{i=1}^n a_{ni}^2 = O(\Delta_n^2)$. If $\sum_{i=1}^n |a_{ni}| = O(1)$, Then*

$$\sum_{i=1}^n \frac{a_{ni}e_i}{\Delta_n} \xrightarrow{D} N(0,1).$$

The proof of the Lemmas 5.5 and 5.6, we can reference the Zhang and Liang (2011)[24] and Zhang and Liang (2013)[19].

Lemma 5.5

- Assumptions (A1), (A2) and (A3), one can imply that $n^{-1}\sum_{i=1}^n \tilde{\xi}_i^2 \to \Sigma_0$, $\max_1|\tilde{\xi}_i| = o(n^{-1/2})$ and $S_n^{-2}\sum_{i=1}^n |\tilde{\xi}_i|$;
- Using (A1), (A2) and (A3), imply that $C_1^{-1}\sum_{i=1}^n \sigma_i^{-2}\tilde{\xi}_{i\ 2}^2$ and $T_n^{-2}\sum_{i=1}^n |\sigma_i^{-2}\tilde{\xi}_i|$;
- Let $\tilde{A}_i = A(t_i) - \sum_{j=1}^n W_{nj}(t_i)A(t_j)$, where $A(\cdot) = f(\cdot), g(\cdot)$ or $h(\cdot)$. Then (A2)(ii) and (A3)(ii) imply that $\max_1|\tilde{A}_i| = o(n^{-1/4})$.

Lemma 5.6 *Under the condition of Lemma 5.5 and (A0), (A3), we have $S_{1n}^2 \to S_n^2 a.s.$*

6. PROOF OF MAIN RESULTS

In the sequel, we use the Abel Inequality (Härdle et al. (2000)[21], page 183). Let $A_1, A_2, \cdots, A_n; B_1, B_2, \cdots, B_n(B_{1_2} \cdots_n 0)$ to be two sequence of real numbers, and $S_k = \sum_{i=1}^k A_i$, $M_1 = \min_1 S_k$, $M_2 = \max_1 S_k$. Then, $B_1 M_1 \sum_{k=1}^n A_k B_{k_1} M_2$. Let $E_i, F_i(1)$ to be arbitrary real numbers and $(j_1, j_2, \cdots, j_n)$ to be a permutation of $(1, \cdots, n)$ such that $F_{j_{1_{j_2}}} \cdots_{j_n}$. Then from the above equation, we have

$$|\sum_{i=1}^n E_iF_i| = |\sum_{i=1}^n E_{j_i}F_{j_i}||\sum_{i=1}^n E_{j_i}(F_{j_i} - F_{j_n})| + |\sum_{i=1}^n E_{j_i}F_{j_n}|$$
$$C\max_1|F_i|\max_1|\sum_{i=1}^m E_{j_i}|. \quad (1)$$

Proof of Theorem 3.1. We prove only *(a)*, as the proof of *(b)* is analogous. From (1) and (??), write that

$$\hat{\beta}_L-\beta=S_{1n}^{-2}[\sum_{i=1}^{n}(\tilde{\xi}_i+\tilde{\mu}_i)(\tilde{y}_i-\tilde{\xi}_i\beta-\tilde{\mu}_i\beta)+n\Xi_\mu^2\beta]$$
$$=S_{1n}^{-2}\{\sum_{i=1}^{n}[(\tilde{\xi}_i+\tilde{\mu}_i)(\tilde{\varepsilon}_i-\tilde{\mu}_i\beta)+\Xi_\mu^2\beta]+\sum_{i=1}^{n}\tilde{\xi}_i\tilde{g}_i+\sum_{i=1}^{n}\tilde{\mu}_i\tilde{g}_i\}$$
$$=S_{1n}^{-2}\{\sum_{i=1}^{n}[(\tilde{\xi}_i+\mu_i)(\varepsilon_i-\mu_i\beta)+\Xi_\mu^2\beta]+\sum_{i=1}^{n}\tilde{\xi}_i\tilde{g}_i+\sum_{i=1}^{n}\tilde{\mu}_i\tilde{g}_i$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{n}W_{nj}(t_i)\tilde{\xi}_i\mu_j\beta-\sum_{i=1}^{n}\sum_{j=1}^{n}W_{nj}(t_i)\tilde{\xi}_i\varepsilon_j-\sum_{i=1}^{n}\sum_{j=1}^{n}W_{nj}(t_i)\varepsilon_i\mu_j$$
$$-\sum_{i=1}^{n}\sum_{j=1}^{n}W_{nj}(t_i)\mu_i\varepsilon_j+2\sum_{i=1}^{n}\sum_{j=1}^{n}W_{nj}(t_i)\mu_i\mu_j\beta$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}W_{nj}(t_i)W_{nk}(t_i)\mu_j\varepsilon_k-\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}W_{nj}(t_i)W_{nk}(t_i)\mu_j\mu_k\beta\}$$
$$:=S_{1n}^{-2}\sum_{l=1}^{10}A_{ln}. \tag{2}$$

Thus. Using Lemma 5.6, in order to prove $S_n^2(\hat{\beta}_L-\beta)/\Sigma_n\xrightarrow{D}N(0,1)$, we verify that

$$\frac{A_{1n}}{\Sigma_n}\xrightarrow{D}N(0,1)\,\frac{A_{kn}}{\Sigma_n}\xrightarrow{P}0fork=2,3,4,5,8,10.\frac{A_{kn}}{\Sigma_n}\xrightarrow{P}0fork=6,7,9.$$

Step 1. we prove that $A_{1n}/\Sigma_n\xrightarrow{D}N(0,1)$.

Set $\omega_i=(\tilde{\xi}_i+\mu_i)(\varepsilon_i-\mu_i\beta)+\Xi_\mu^2\beta$ and $Z_{ni}=\omega_i/\Sigma_n$. According to Zhang and Liang (2013)[19] we have Σ_n^2Cn. Using (A0), Lemma 5.5, Σ_n^2Cn. We deduce that $EZ_{ni}=0,Var(\sum_{i=1}^{n}Z_{ni})=1$ and $E|Z_{ni}|^{2+\delta}<\infty$. Owing to $\{\varepsilon_i\}$ are sequence of zero mean stationary NA random variables, $\{e_i\sigma_i-\mu_i\beta\}$ are also sequence of zero mean stationary NA random variables, $\{\tilde{\xi}_i+\mu_i\}$ are sequence of i.i.d. random variables. Using Definition (1), we know $(\tilde{\xi}_i+\mu_i)(\varepsilon_i-\mu_i\beta)$ are sequence of ρ-mixing random variables, and the mixing coefficients $\rho(n)=0$. In this situation, we can know ρ-mixing is also a squence of strong mixing from Fan and Yao (2003)[25], and we have $0\alpha(n)\rho(n)/4=0$. Therefore, $(\tilde{\xi}_i+\mu_i)(\varepsilon_i-\mu_i\beta)$ are sequences of strong mixing random variables with the mixing coefficients $\alpha(n)=0$. Thus. By the proof of Theorem 3.1 of Zhang and Liang (2013)[19], we accept the conclusion as correct.

Step 2. We prove that $A_{kn}/\Sigma_n\to0$ for $k=2,3,4,5,8,10$.

From (A0)(i), (A3) and Lemma 5.2, we can verity that

$$\sum_{i=1}^{n}(\zeta_i-E\zeta_i)=O(n^{\frac{1}{2}}logn)a.s.\max_{1}|\sum_{j=1}^{n}W_{nj}(t_i)\mu_j|=O(n^{-\frac{1}{4}}logn)a.s. \tag{3}$$

where $\zeta_i=|\mu_i|,\mu_i^2or\mu_i$.

Since the $\{\varepsilon_i,i=1,2,\ldots,n\}$ are sequence of zero mean stationary NA random errors, the $\{\varepsilon_i^+,i=1,2,\ldots n\}$ and $\{\varepsilon_i^-,i=1,2,\ldots n\}$ are all NA sequence. From Lemma 5.3, one can get $1/n\sum_{i=1}^{n}\varepsilon_i^+=o(1)a.s.,1/n\sum_{i=1}^{n}\varepsilon_i^-=o(1)a.s.$ And $|\varepsilon_i|=\varepsilon_i^++\varepsilon_i^-$, we have

$$\frac{1}{n}\sum_{i=1}^{n}|\varepsilon_i|=o(1)a.s. \tag{4}$$

Hence, by applying (A0)(i) and (A3), Lemma 5.1 and $a_n=W_{nj}(t_i),\alpha=4$, one can obtain that

$$\max_{1}|\sum_{j=1}^{n}W_{nj}(t_i)\varepsilon_j|=o(n^{-\frac{1}{4}})a.s. \tag{5}$$

So. From (A0)(i), (A1), (A2), (A3), Lemma 5.5, (??), (3), (5) we deduce that

$$|\frac{A_{2n}}{\Sigma_n}|\frac{C}{\sqrt{n}}|\sum_{i=1}^{n}\tilde{\xi}_i\tilde{g}_i|\frac{C}{\sqrt{n}}\{|\sum_{i=1}^{n}\tilde{h}_i\tilde{g}_i|+|\sum_{i=1}^{n}v_i\tilde{g}_i|+|\sum_{i=1}^{n}[\sum_{j=1}^{n}W_{nj}(t_i)v_j]\tilde{g}_i|\}$$
$$\frac{C}{\sqrt{n}}[n\cdot\max_{1}|\tilde{h}_i|\cdot\max_{1}|\tilde{g}_i|+\max_{1}|\tilde{g}_i|\cdot\max_{1}|\sum_{i=1}^{n}v_{k_i}|$$
$$+\max_{1}\sum_{i=1}^{n}W_{nj}(t_i)\cdot\max_{1}|\tilde{g}_i|\cdot\max_{1}|\sum_{j=1}^{n}v_{k_j}|]$$
$$=o(1)+o(n^{-1/4}\log n)=o(1).$$
$$E(\frac{A_{3n}}{\Sigma_n})^2\frac{C}{n}\{E(\sum_{i=1}^{n}\tilde{g}_i\mu_i)^2+E[\sum_{i=1}^{n}\tilde{g}_i\sum_{j=1}^{n}W_{nj}(t_i)\mu_j]^2\}$$
$$=\frac{C}{n}\{\sum_{i=1}^{n}\tilde{g}_i^2+\sum_{j=1}^{n}[\sum_{i=1}^{n}W_{nj}(t_i)\tilde{g}_i]^2\}=o(n^{-1/2})$$
$$A_{4n}=\sum_{i=1}^{n}\tilde{h}_i\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\beta+\sum_{i=1}^{n}v_i\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\beta-\sum_{i=1}^{n}\sum_{s=1}^{n}W_{ns}(t_i)v_s\sum_{j=1}^{n}W_{nj}(t_i)\mu_j\beta$$
$$:=D_{1n}+D_{2n}+D_{3n}.$$
$$E(\frac{D_{1n}}{\Sigma_n})^2\frac{C}{n}\sum_{j=1}^{n}[\sum_{i=1}^{n}W_{nj}(t_i)\tilde{h}_i]^2\cdot\max_{1}|\tilde{h}_i|^2\cdot\max_{1}|\sum_{i=1}^{n}W_{nj}(t_i)|^2=o(n^{-1/2}).$$
$$E(\frac{D_{2n}}{\Sigma_n})^2\frac{C}{n}\sum_{j=1}^{n}[\sum_{i=1}^{n}W_{nj}(t_i)v_i]^2\cdot\max_{1}|\sum_{i=1}^{m}v_{k_i}|^2\cdot\max_{1,j}W_{nj}^2(t_i)=o(1).$$
$$\frac{|D_{3n}|}{\Sigma_n}\frac{C}{\sqrt{n}}\max_{1}|\sum_{s=1}^{m}v_{k_s}|\cdot\max_{1}\sum_{i=1}^{n}W_{ns}(t_i)\cdot\max_{1}|\sum_{j=1}^{n}W_{nj}(t_i)\mu_j|=o_p(1).$$

$\frac{|A_{5n}|}{\Sigma_n}\frac{C}{\sqrt{n}}\sum_{i=1}^{n}\tilde{\xi}_i\sum_{j=1}W_{nj}(t_i)\varepsilon_j$
$\frac{C}{\sqrt{n}}(\sum_{i=1}^{n}\tilde{h}_i\sum_{j=1}^{n}W_{nj}(t_i)\varepsilon_j+\sum_{i=1}^{n}v_i\sum_{j=1}^{n}W_{nj}(t_i)\varepsilon_j+\sum_{i=1}^{n}\sum_{j=1}^{n}W_{nj}(t_i)v_j\sum_{i=1}^{n}W_{nj}(t_i)\varepsilon_j)$
$\frac{C}{\sqrt{n}}(\max_1|\tilde{h}_i|\max_1|\sum_{j=1}^{n}W_{nj}(t_i)\varepsilon_j|+\max_1|\sum_{i=1}^{n}v_{ki}|\max_1|\sum_{j=1}^{n}W_{nj}(t_i)\varepsilon_j|$
$+\max_1\sum_{i=1}^{n}W_{nj}(t_i)\max_1|\sum_{j=1}^{n}v_{kj}|\max_1|\sum_{i=1}^{n}W_{nj}(t_i)\varepsilon_j|)$
$=o(1)$
$E(\frac{A_{8n}}{\Sigma_n})^2\frac{C}{n}\cdot\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}W_{nj_1}(t_{i_1})W_{nj_2}(t_{i_2})E(\mu_{i_1}\mu_{i_2}\mu_{j_1}\mu_{j_2})$
$\frac{C}{n}\cdot[\sum_{i=1}^{n}\sum_{j=1}^{n}W_{nj}^2(t_i)E(\mu_i^2\mu_j^2)+\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}W_{ni_1}(t_{i_1})W_{ni_2}(t_{i_2})E(\mu_{i_1}^2\mu_{i_2}^2)]$
$=O(n^{-1}).$
$E(\frac{A_{10n}}{\Sigma_n})^2\frac{C}{n}\cdot$
$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}W_{nj_1}(t_{i_1})W_{nj_2}(t_{i_2})W_{nk_1}(t_{i_1})W_{nk_2}(t_{i_2})E(\mu_{j_1}\mu_{j_2}\mu_{k_1}\mu_{k_2})$
$\frac{C}{n}\cdot[\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}W_{nj}(t_{i_1})W_{nj}(t_{i_2})W_{nk}(t_{i_1})W_{nk}(t_{i_2})E(\mu_j^2\mu_k^2)$
$+\sum_{i=1}^{n}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\sum_{k=1}^{n}W_{nj_1}(t_i)W_{nj_2}(t_i)W_{nk}^2(t_i)E(\mu_{j_1}\mu_{j_2}\mu_k^2)$
$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}W_{nj}^2(t_i)W_{nk_1}(t_i)W_{nk_2}(t_i)E(\mu_j^2\mu_{k_1}\mu_{k_2})]$
$=O(n^{-1}).$

Step 3. We prove that $A_{kn}/\Sigma_n\to 0$ for $k=6,7,9$.

Since $\{\varepsilon_i\}$ are sequence of zero mean stationary NA random variables, $Cov(\varepsilon_{i_1},\varepsilon_{i_2})0$ for $i_1\neq i_2$. From (A0)(i), (A3), we can get

$E(\frac{A_{6n}}{\Sigma_n})^2=E[\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}W_{nj_1}(t_{i_1})W_{nj_2}(t_{i_2})\varepsilon_{i_1}\varepsilon_{i_2}\mu_{j_1}\mu_{j_2}]$
$=\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{j=1}^{n}W_{nj}(t_{i_1})W_{nj}(t_{i_2})E\varepsilon_{i_1}\varepsilon_{i_2}E\mu_j^2$
$=\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{j=1}^{n}W_{nj}(t_{i_1})W_{nj}(t_{i_2})Cov(\varepsilon_{i_1},\varepsilon_{i_2})\cdot E\mu_j^2$
$\sum_{i=1}^{n}\sum_{j=1}^{n}W_{nj}^2(t_i)Cov(\varepsilon_i,\varepsilon_i)E\mu_j^2$
$C\sum_{i=1}^{n}\sum_{j=1}^{n}W_{nj}^2(t_i)=O(1).$
$E(\frac{A_{7n}}{\Sigma_n})^2=E[\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}W_{nj_1}(t_{i_1})W_{nj_2}(t_{i_2})\varepsilon_{j_1}\varepsilon_{j_2}\mu_{i_1}\mu_{i_2}]$
$=\sum_{i=1}^{n}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}W_{nj_1}(t_i)W_{nj_2}(t_i)E\varepsilon_{j_1}\varepsilon_{j_2}E\mu_i^2$
$=\sum_{i=1}^{n}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}W_{nj_1}(t_i)W_{nj_2}(t_i)Cov(\varepsilon_{j_1},\varepsilon_{j_2})E\mu_i^2$
$\sum_{i=1}^{n}\sum_{j=1}^{n}W_{nj}^2(t_i)Cov(\varepsilon_j,\varepsilon_j)E\mu_i^2$
$C\sum_{i=1}^{n}\sum_{j=1}^{n}W_{nj}^2(t_i)=O(1).$
$E(\frac{A_{9n}}{\Sigma_n})^2=$
$E[\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}W_{nj_1}(t_{i_1})W_{nj_2}(t_{i_2})W_{nk_1}(t_{i_1})W_{nk_2}(t_{i_2})\varepsilon_{k_1}\varepsilon_{k_2}\mu_{j_1}\mu_{j_2}]$
$=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}W_{nj}(t_i)^2W_{nk_1}(t_{i_1})W_{nk_2}(t_{i_2})E\varepsilon_{k_1}\varepsilon_{k_2}E\mu_j^2$
$=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}W_{nj}(t_i)^2W_{nk_1}(t_{i_1})W_{nk_2}(t_{i_2})Cov(\varepsilon_{k_1},\varepsilon_{k_2})E\mu_j^2$
$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}W_{nj}(t_i)^2W_{nk_1}(t_{i_1})W_{nk_2}(t_{i_2})Cov(\varepsilon_k,\varepsilon_k)E\mu_j^2$
$C\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k_1=1}^{n}\sum_{k_2=1}^{n}W_{nj}(t_i)^2W_{nk_1}(t_{i_1})W_{nk_2}(t_{i_2})=O(1).$

Thus, the proof of Theorem (3.1) is completed.

Proof of Theorem 3.2. We prove only *(a)*, the proof for *(b)* is similar. From (5), note that

$\hat{g}_L(t)-E\hat{g}_L(t)=\sum_{i=1}^{n}W_{ni}(t)[y_i-x_i\hat{\beta}_L-Ey_i+E(x_i\hat{\beta}_L)]$
$=\sum_{i=1}^{n}W_{ni}(t)(\varepsilon_i-\mu_i\beta)+\sum_{i=1}^{n}W_{ni}(t)\xi_i(\beta-\hat{\beta}_L)$
$-\sum_{i=1}^{n}W_{ni}(t)\xi_iE(\beta-\hat{\beta}_L)+\sum_{i=1}^{n}W_{ni}(t)\mu_i(\beta-\hat{\beta}_L)$
$-\sum_{i=1}^{n}W_{ni}(t)E[\mu_i(\beta-\hat{\beta}_L)]$
$:=F_{1n}(t)+F_{2n}(t)-F_{3n}(t)+F_{4n}(t)-F_{5n}(t).$

Therefore, we only need to prove that

$$F_{1n}(t)/\Gamma_n(t)\xrightarrow{D}N(0,1),F_{sn}(t)=O_p(n^{-1/2})fors=2,4,F_{kn}(t)=O(n^{-1/2})fork=3,5.$$

Step 1. we prove that $F_{1n}(t)/\Gamma_n(t) \xrightarrow{D} N(0,1)$.

Since $\{\varepsilon_i - \mu_i\beta\}$ are sequence of zero mean stationary NA random variables. From (A0)(i), (A3), one can achieve that $\max_1|W_{ni}(t)| = o(\Gamma_n(t))$ and $\sum_{i=1}^n |W_{ni}(t)| = O(1)$. Note that $n\Gamma_n^2(t) \to \infty$ together with the conditions in Theorem 3.2, using Lemma 5.4, we obtain that

$$F_{1n}(t)/\Gamma_n(t) \xrightarrow{D} N(0,1).$$

Step 2. we prove that $F_{sn}(t) = O_p(n^{-1/2}) fors = 2,4$.

Firstly, We prove that $E(\hat{\beta}_L - \beta) = O(n^{-1})$. Taking the same notations of A_{kn} for $k = 1,2,\cdots,10.$ as the proof of Theorem 3.1. Observe that

$$E[S_{1n}^2(\hat{\beta}_L - \beta)]^2 = E[\sum_{k=1}^{10} A_{kn}]^2 C \cdot \sum_{k=1}^{10} E(A_{kn})^2.$$

Applying (A0)(i), (A2),(A3), Lemma 5.5, Lemma 5.6 and (3), (5) Noticing that $\{\varepsilon_i - \mu_i\beta\}$ are sequences of NA variables and $\{\varepsilon_i\mu_i\}$ are sequences of α-mixing variables, one can achieve that

$$\sup_n n^{-1}E(A_{1n})^2 \sup_n Cn^{-1}\{E[\sum_{i=1}^n \tilde{\xi}_i(\varepsilon_i - \mu_i\beta)]^2 + E(\sum_{i=1}^n \varepsilon_i\mu_i)^2 + \sum_{i=1}^n E(\mu_i^2 - \Xi_\mu^2)^2\beta^2\}$$
$$:= \sup_n Cn^{-1}(E(A_{11n})^2 + E(A_{12n})^2 + E(A_{13n})^2)$$
$$\sup_n n^{-1}E(A_{11n})^2 \sup_n \frac{C}{n} E(\sum_{i=1}^n \tilde{\xi}_i^2\varepsilon_i^2 + \sum_{i=1}^n \tilde{\xi}_i^2\mu_i^2\beta^2 - 2\sum_{i=1}^n \tilde{\xi}_i^2\varepsilon_i\mu_i\beta)$$
$$\sup_n \frac{C}{n}(\sum_{i=1}^n \tilde{\xi}_i^2 E\varepsilon_i^2 + \sum_{i=1}^n \tilde{\xi}_i^2 E\mu_i^2\beta^2 + 2E\sum_{i=1}^n |\tilde{\xi}_i^2\varepsilon_i\mu_i\beta|)$$
$$\sup_n \frac{C}{n}(O(n) + O(n) + \sqrt{\sum_{i=1}^n \tilde{\xi}_i^2 E\varepsilon_i^2 \cdot \sum_{i=1}^n \tilde{\xi}_i^2 E\mu_i^2\beta^2}) < \infty$$
$$\sup_n n^{-1}E(A_{12n})^2 = \sup_n \frac{1}{n} E(\sum_{i=1}^n \varepsilon_i\mu_i)^2 \sup_n \frac{C}{n}(\sum_{i=1}^n \sigma_i^2\Xi_\mu^2) < \infty$$
$$\sup_n n^{-1}E(A_{13n})^2 = \sup_n \frac{1}{n}\sum_{i=1}^n E(\mu_i^2 - \Xi_\mu^2)\beta^2 \sup_n \frac{C}{n} n < \infty$$
$$\sup_n n^{-1}E(A_{6n})^2 = \sup_n \frac{1}{n} E\{\sum_{j=1}^n [\sum_{i=1}^n W_{nj}(t_i)\varepsilon_i]\mu_j\}^2 = \sup_n \frac{C}{n}\{\sum_{j=1}^n E[\sum_{i=1}^n W_{nj}(t_i)\varepsilon_i]^2 E\mu_j^2\}$$
$$\sup_n \frac{C}{n}[\sum_{j=1}^n \sum_{i=1}^n W_{nj}^2(t_i) E(\sum_{i=1}^n \varepsilon_i^2)\Xi_\mu^2] C \cdot \sum_{j=1}^n \sum_{i=1}^n W_{nj}^2(t_i)$$
$$C \max_1 \sum_{i=1}^n W_{nj}(t_i) \max_1 \sum_{j=1}^n W_{nj}(t_i) < \infty$$

Similarly, one can deduce that $\sup_n n^{-1} \cdot E(A_{kn})^2 < \infty$ for $k = 2,3,4,5,7,8,9,10$. Bying (A0)(i), (A2), (A3), \

Lemmas 5.5, Lemma 5.6 and (3), (5). Therefore, from Lemma 5.5, Lemma 5.6 one can deduce that

$$E(\hat{\beta}_L - \beta)^2 = O(n^{-1}) E|\hat{\beta}_L - \beta| = O(n^{-1/2}) \tag{6}$$

So, from (A0)(i), (A2), (A3), Lemma 5.5, (3), (5) and (6), we can get

$$E|F_{2n}(t)| E|\beta - \hat{\beta}_L| \cdot \max_1|\xi_i| \cdot \sum_{i=1}^n W_{ni}(t) = o(n^{-1})$$
$$|F_{3n}(t)| E|\beta - \hat{\beta}_L| \cdot \max_1|\xi_i| \cdot \sum_{i=1}^n W_{ni}(t) = o(n^{-1})$$
$$E|F_{4n}(t)| E|\beta - \hat{\beta}_L| \cdot \max_1|\sum_{i=1}^n W_{ni}(t)\mu_i| = o(n^{-\frac{3}{4}})$$
$$|F_{5n}(5)| \max_1 \sum_{i=1}^n W_{ni}(t) \cdot \sqrt{E\mu_i^2 E(\beta - \hat{\beta}_L)^2} = O(n^{-\frac{1}{2}})$$

Thus, the proof of Theorem 3.2 is completed.

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