GLOBAL EXISTENCE AND BLOW-UP FOR NONLINEAR PARABOLIC EQUATION WITH NONLINEAR NEUMANN BOUNDARY CONDITION

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ABSTRACT
This article deal with the global existence and blow-up time of solution for the nonlinear parabolic equation under the nonlinear Neumann boundary condition. we establish, blow-up will occur at some finite time and an upper bound time. Moreover, we also show a lower bound time.

Keywords: *Global existence; Blow-up; Nonlinear Neumann boundary condition.*

1. INTRODUCTION

In this paper, we main study of the nonlinear parabolic equation under the nonlinear boundary conditions problem

$$\begin{cases} u_t = div\left[|\nabla u|^p \nabla u\right] - f(u), & (x,t) \in \Omega \times (0,t^*), \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial n} = g(u), & (x,t) \in \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) > 0, & x \in \Omega, \end{cases} \qquad (1)$$

where $p \geq 2$, $\Omega \in R^N$ $(N \geq 2)$ is a bounded domain with smooth boundary, $\frac{\partial \mathrm{u}}{\partial n}$ is the outward normal derivative of u on the boundary $\partial\Omega$, t^* is the blow-up time if blow-up occurs, or else $t^* = \infty$, $u_0(x)$ is nonnegative and continuous , also $u_0(x) \in C^1(\overline{\Omega})$ $(u_0(x) \leq M_1)$.

This problem (1) can be use to explain the phenomenon of many physical and biological, the nonlinear Neumann boundary conditions nonlinear equation(see[1,2]). we derive some result in the paper. In the last few years,a lot of works have been devoted to the deal with the solutions to nonlinear parabolic equations (one can see[3-6]). we refer the reader to[7-10]and the references therein. In [11], Mu et al. investigated the following initial boundary value problem

$$\begin{cases} u_t = \mathrm{div}\left(|\nabla u|^{r-2} \nabla u\right) - f(u), & (x,t) \in \Omega \times (0,t^*) \\ |\nabla u|^{r-2} \dfrac{\partial u}{\partial n} = g(u), & (x,t) \in \partial\Omega \times (0,t^*) \\ u(x,0) = u_0(x) > 0, & x \in \Omega \end{cases} \qquad (2)$$

where $r \geq 2$, they esablish sufficient condition the solution $u(x,t)$ exists globally and an upper bound establish for the blow up in finite time t^*. Recently, Fang and Chai[12] studied the quasilinear parabolic equation problem

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$$\begin{cases} u_t = \left[\left(|\nabla u|^p + 1\right)u_{,i}\right]_i - f(u), & (x,t) \in \Omega \times (0, t^*) \\ \left(|\nabla u|^p + 1\right)\dfrac{\partial u}{\partial \nu} = g(u), & (x,t) \in \partial\Omega \times (0, t^*) \\ u(x,0) = u_0(x) > 0, & x \in \overline{\Omega} \end{cases} \quad (3)$$

They by using the suitable techniques of differential inequalities, derived blow-up occurs upper and lower bounds of the blow-up time.The rest of this paper is organized as follows.

2. LEMMA

Assume the solution $u(x,t)$ of the problem (1) exists in $(0, T_0)$ for some $T_0 > 0$ and $u_0'(x) \le M_1$,for $0 \le x \le 1$. Then $|\nabla u| \le M_1$.

Proof. The technique for proving the maximum principle is quite standard. Reader may refer to Lemma2.1 by Y.Yang,et.al [13] omit it here.

3. THE GLOBAL EXISITENCE

In this section, we establish the sufficient condition on the function f and g , which ensure that $u(x,t)$ exists globally. we get the following result.

Theorem1. Let $u(x,t)$ be the solution of problem (1) and assume that the nonnegative functions f and g satisfy the following conditions $f(\xi) \ge k_1 \xi^r$, $\xi \ge 0$ and $g(\xi) \le k_2 \xi^q$, $\xi \ge 0$, (4)

where $k_1 \ge 0$, $k_2 \ge 0$, $q > 1$ and $q - 1 < p < r - 1$. Then the nonnegative solution $u(x,t)$ of problem (1) exists globally for all time $t > 0$.

Proof. We set $\Phi(t) = \int_\Omega u^2 dx$ (5)

Taking the derivative of $\Phi(t)$ with respect to t, then it follows from (1) and (4), we have

$$\begin{aligned} \Phi'(t) &= 2\int_\Omega u u_t dx = 2\int_{\partial\Omega} |\nabla u|^2 u g(u) ds - 2\int_\Omega |\nabla u|^{p+2} dx - 2\int_\Omega u f(u) dx \\ &\le 2k_2 M_1 \int_{\partial\Omega} u^{q+1} ds - 2\int_\Omega |\nabla u|^{p+2} dx - 2k_1 \int_\Omega u^{r+1} dx \end{aligned} \quad (6)$$

By Lemma2.1 in [12], we get $\int_{\partial\Omega} u^{q+1} ds \le \dfrac{N}{\rho_0} \int_\Omega u^{q+1} dx + \dfrac{(q+1)d}{\rho_0} \int_\Omega u^q |\nabla u| dx,$ (7)

Where $\rho_0 = \min_{x \in \partial\Omega} (x \cdot n) > 0$, $d = \max_{x \in \partial\Omega} |x|$. Combining (6) and (7), we obtain

$$\Phi'(t) \le \frac{2k_2 M_1 N}{\rho_0} \int_\Omega u^{q+1} dx + \frac{2k_2 M_1 (q+1)d}{\rho_0} \int_\Omega u^q |\nabla u| dx - 2\int_\Omega |\nabla u|^{p+2} dx - 2k_1 \int_\Omega u^{r+1} dx \quad (8)$$

By using Young's inequality, we have,

$$\int_\Omega u^q |\nabla u| dx \le \frac{(p+1)\varepsilon}{p+2} \int_\Omega u^{\frac{(p+2)q}{p+1}} dx + \frac{1}{(p+2)\varepsilon} \int_\Omega |\nabla u|^{p+2} dx, \quad (9)$$

Where $\varepsilon > 0$. Combining (8) and (9), we have

$$\Phi'(t)\le \frac{2k_2M_1N}{\rho_0}\int_\Omega u^{q+1}dx+\frac{2k_2M_1(q+1)(p+1)d\varepsilon}{\rho_0(p+2)}\int_\Omega u^{\frac{(p+2)q}{p+1}}dx$$
$$-\left[2-\frac{2k_2M_1(q+1)d}{\rho_0(p+2)\varepsilon}\right]\int_\Omega|\nabla u|^{p+2}dx-2k_1\int_\Omega u^{r+1}dx \tag{10}$$

Since $\left|\nabla u^{\frac{p}{2}+1}\right|^2=\left(\frac{p}{2}+1\right)^2u^p|\nabla u|^2$, It follows from Hölder inequality that

$$\int_\Omega\left|\nabla u^{\frac{p}{2}+1}\right|^2dx\le\left(\frac{p}{2}+1\right)^2\left(\int_\Omega|\nabla u|^{p+2}dx\right)^{\frac{2}{p+2}}\cdot\left(\int_\Omega u^{p+2}dx\right)^{\frac{p}{p+2}}, \tag{11}$$

By membrane inequality $\lambda_1\int_\Omega\varpi^2dx\le\int_\Omega|\nabla\varpi|^2dx$ in[13]. where λ_1 is the first eigenvalue in the fixed membrane problem $\Delta\varpi+\lambda\varpi=0$, $\varpi>0$ in Ω, $\varpi=0$ on $\partial\Omega$.

And it follows from upper, we have $\int_\Omega u^{p+2}dx\le\left[\frac{(p+2)^2}{4\lambda_1}\right]^{\frac{p}{2}+1}\cdot\int_\Omega|\nabla u|^{p+2}dx$, (12)

By Hölder inequality, we have $\int_\Omega u^{\frac{(p+2)q}{p+1}}dx\le\left(\int_\Omega u^{q+1}dx\right)^\alpha\cdot\left(\int_\Omega u^{r+1}dx\right)^{1-\alpha}$, (13)

with $\alpha=\frac{(p+2)q-(p+1)(r+1)}{(p+1)(q-r)}$, We making use of the following inequality

$$a_1^{s_1}a_2^{s_2}\le s_1a_1+s_2a_2,\ a_1,a_2>0,\ s_1,s_2>0 \text{ and } s_1+s_2=1. \tag{14}$$

It follows from (15), we have

$$\int_\Omega u^{\frac{(p+2)q}{p+1}}dx\le\left(\kappa^{\frac{\alpha-1}{\alpha}}\int_\Omega u^{q+1}dx\right)^\alpha\cdot\left(\kappa\int_\Omega u^{r+1}dx\right)^{1-\alpha}\le\alpha\kappa^{\frac{\alpha-1}{\alpha}}\int_\Omega u^{q+1}dx+(1-\alpha)\kappa\int_\Omega u^{r+1}dx \tag{15}$$

for $\kappa>0$ to be confirmed. By inserting(13), (15) in (11), we obtain(16) as follows

$$\Phi'(t)\le\left\{\left(\frac{2k_2M_1N}{\rho_0}+\frac{2k_2M_1(q+1)(p+1)d\varepsilon\alpha}{\rho_0(p+2)}\kappa^{\frac{\alpha-1}{\alpha}}\right)\int_\Omega u^{q+1}dx-2\left[\frac{4\lambda_1}{(p+2)^2}\right]^{\frac{p}{2}+1}\int_\Omega u^{p+2}dx\right\}$$
$$+\left\{\frac{2k_2M_1(q+1)d}{\rho_0(p+2)\varepsilon}\cdot\left[\frac{4\lambda_1}{(p+2)^2}\right]^{\frac{p}{2}+1}\int_\Omega u^{p+2}dx-\left[2k_1-\frac{2k_2M_1(q+1)(p+1)d\varepsilon}{\rho_0(p+2)}(1-\alpha)\kappa\right]\int_\Omega u^{r+1}dx\right\}$$ B
$$=I_1+I_2$$

y Hölder inequality, we have

$$\int_\Omega u^{q+1}dx\le\left(\int_\Omega u^{p+2}dx\right)^{\frac{q+1}{p+2}}\cdot|\Omega|^{\frac{p-q+1}{p+2}} \text{ and } \Phi(t)=\int_\Omega u^2dx\le\left(\int_\Omega u^{q+1}dx\right)^{\frac{2}{q+1}}\cdot|\Omega|^{\frac{q-1}{q+1}} \tag{17}$$

Combining (16)and(17), we have $I_1\le\int_\Omega u^{q+1}dx\left[M_1-M_2\Phi^{\frac{p-q+1}{2}}(t)\right]$, (18) where $$M_1=\frac{2k_2M_1N}{\rho_0}+\frac{2k_2M_1(q+1)(p+1)d\varepsilon\alpha}{\rho_0(p+2)}\kappa^{\frac{\alpha-1}{\alpha}}>0\ , \tag{19}$$

$$M_2=2\left[\frac{4\lambda_1}{(P+2)^2}\right]^{\frac{P}{2}+1}\cdot|\Omega|^{-\frac{p-q+1}{2}}>0\ , \tag{20}$$

Similarly to, using again Hölder inequality, we have

$$\int_\Omega u^{p+2}dx\le\left(\int_\Omega u^{r+1}dx\right)^{\frac{p+2}{r+1}}\cdot|\Omega|^{\frac{r-p-1}{r+1}}\ \text{ and }\ \Phi(t)=\int_\Omega u^2dx\le\left(\int_\Omega u^{r+1}dx\right)^{\frac{2}{r+1}}\cdot|\Omega|^{\frac{r-1}{r+1}}\ , \tag{21}$$

Combining (16) and (21), we have $$I_2\le\left(\int_\Omega u^{r+1}dx\right)^{\frac{p+2}{r+1}}\left[M_3-M_4\Phi^{\frac{r-p-1}{2}}(t)\right], \tag{22}$$

where $$M_3=\frac{2k_2M_1(q+1)d}{\rho_0(p+2)\varepsilon}\cdot\left[\frac{4\lambda_1}{(P+2)^2}\right]^{\frac{P}{2}+1}>0, \tag{23}$$

$$M_4=\left[2k_1-\frac{2k_1M_1(q+1)(p+1)d\varepsilon}{\rho_0(p+2)}(1-\alpha)\kappa\right]\cdot|\Omega|^{\frac{(1-r)(r-p-1)}{2(r+1)}}\ , \tag{24}$$

We can choose κ small enough to $M_4>0$.

$$\Phi'(t)\le\int_\Omega u^{q+1}dx\left[M_1-M_2\Phi^{\frac{p-q+1}{2}}(t)\right]+\left(\int_\Omega u^{r+1}dx\right)^{\frac{p+2}{r+1}}\left[M_3-M_4\Phi^{\frac{r-p-1}{2}}(t)\right], \tag{25}$$

Hence, from the type we can get the following conclusion

(1). If $\Phi(t)<\min\left\{\left(\frac{M_1}{M_2}\right)^{\frac{2}{p-q+1}},\left(\frac{M_3}{M_4}\right)^{\frac{2}{r-p-1}}\right\}$,It is true. That is, $\Phi(t)$ remains bounded.

(2). If $\Phi(t)\ge\max\left\{\left(\frac{M_1}{M_2}\right)^{\frac{2}{p-q+1}},\left(\frac{M_3}{M_4}\right)^{\frac{2}{r-p-1}}\right\}$, $\Phi(t)$ is decreasing in each time interval. we have $\Phi(t)<\Phi(0)$. So that $\Phi(t)$ remains bounded for all time in Theorem1. This completes the proof.

4. LOWER BOUNDS FOR t^*

In this section, we seek a lower bound for the blow-up time t^*, assuming that Ω is a star shaped domain, we establish the result as follows.

Theorem3. Let $u(x,t)$ be the nonnegative solution of problem (1) and $u(x,t)$ blow up at t^*, and that the data f and g satisfy the conditions

$$f(\xi)\ge k_1\xi^r,\ \ \xi\ge0\ \text{ and }\ g(\xi)\le k_2\xi^{1+\frac{n}{2}},\ \ \xi\ge0 \tag{26}$$

with $k_1>0$, $k_2>0$, $r>5$, $n>4$. we defined the auxiliary function $\varphi(t):=\int_\Omega u^{2n}dx$, (27)

and show that $\varphi(t)$ satisfies inequality $\varphi'(t)\le\Gamma(\varphi)$, (28)

for some function $\Gamma(\varphi)$. It follows that t^* is bounded from below by $t^*\ge\int_{\varphi(0)}^{\infty}\frac{d\eta}{\Gamma(\eta)}d\eta$. (29)

Proof. Taking the derivative of $\varphi(t)$ with respect to t, making use of the boundary condition together with the condition (27), we obtain (30) as follows

$$\varphi'(t) \le 2nkM_1 \int_{\partial\Omega} u^{\frac{5}{2}n} ds - 2n(2n-1)\int_\Omega u^{2n-2}|\nabla u|^{p+2} dx - 2n(2n-1)\int_\Omega u^{2n-2}|\nabla u|^2 dx - 2nk_1 \int_\Omega u^{2n+r-1} dx$$

By using of (7), we have

$$\int_{\partial\Omega} u^{\frac{5}{2}n} ds \le \frac{N}{\rho_0}\int_\Omega u^{\frac{5}{2}n} dx + \frac{\frac{5}{2}nd}{\rho_0}\int_\Omega u^{\frac{5}{2}n-1}|\nabla u| dx \qquad (31)$$

Substituting (31) into (30), we have

$$\begin{aligned}\varphi'(t) \le & \frac{2nk_2M_1N}{\rho_0}\int_\Omega u^{\frac{5}{2}n} dx + \frac{5n^2k_2M_1d}{\rho_0}\int_\Omega u^{\frac{5}{2}n-1}|\nabla u| dx - 2n(2n-1)\int_\Omega u^{2n-2}|\nabla u|^{p+2} dx \\ & -2n(2n-1)\int_\Omega u^{2n-2}|\nabla u|^2 dx - 2nk_1\int_\Omega u^{2n+r-1} dx\end{aligned} \qquad (32)$$

Making use of Young inequality, we have $\int_\Omega u^{\frac{5}{2}n-1}|\nabla u| dx \le \frac{\mu}{2}\int_\Omega u^{2n-2}|\nabla u|^2 + \frac{1}{2\mu}\int_\Omega u^{3n} dx$, (33)

For all $\mu > 0$, we choose μ small enough such that $\frac{5n^2k_2M_1d\mu}{2\rho_0} - 2n(2n-1) = 0$, By Young inequality, we have

$$\int_\Omega u^{\frac{5}{2}n} dx \le \left(\int_\Omega u^{2n} dx \cdot \int_\Omega u^{3n} dx\right)^{\frac{1}{2}} \le \frac{1}{2}\int_\Omega u^{2n} dx + \frac{1}{2}\int_\Omega u^{3n} dx, \qquad (34)$$

Combining (32),(33) and (34), we obtain (35) as follows

$$\varphi'(t) \le \frac{nk_2M_1N}{\rho_0}\varphi(t) + \left(\frac{nk_2M_1N}{\rho_0} + \frac{5n^2k_2M_1d}{2\mu\rho_0}\right)J_1(t) - 2n(2n-1)J_2(t) - 2nk_1\int_\Omega u^{2n+r-1} dx$$

Using Sobolev type inequality derived by Payne et al. [10] and from (35) we have

$$J_1(t) = \int_\Omega u^{3n} dx \le \left[\frac{3}{\rho_0}\int_\Omega u^{2n} dx + \frac{n(\rho_0+d)}{\rho_0}\int_\Omega u^{2n-1}|\nabla u| dx\right]^{\frac{3}{2}}, \qquad (36)$$

By using Hölder inequality, we have $\int_\Omega u^{2n-1}|\nabla u| dx \le \left(\int_\Omega u^\delta dx\right)^{\frac{p+1}{p+2}} \cdot J_2^{\frac{1}{P+2}}(t)$, (37)

where $\delta = \frac{(p+2)(2n-1)-2(n-1)}{(p+1)}$, Again using Hölder inequality, we get(38) as follows

$\int_\Omega u^\delta dx \le \varphi^{\frac{\delta}{2n}}(t)\cdot|\Omega|^{1-\frac{\delta}{2n}}$. Combining (36), (37) and (38), we have

$$J_1(t) \le \left[\frac{3}{\rho_0}\varphi(t) + \frac{n(\rho_0+d)}{\rho_0}\cdot|\Omega|^{\frac{(2n-\delta)(p+1)}{2n(p+2)}}\varphi^{\frac{\delta(p+1)}{2n(p+2)}}(t)J_2^{\frac{1}{p+2}}(t)\right]^{\frac{3}{2}}, \qquad (39)$$

we make use of the following inequality $(a_1+a_2)^{\frac{3}{2}} \le \sqrt{2}\left(a_1^{\frac{3}{2}} + a_2^{\frac{3}{2}}\right)$, (40)

Combining (39) and (40), we have $J_1(t) \le \tilde{c}_1(t)\varphi^{\frac{3}{2}}(t) + \tilde{c}_2(t)\varphi^{\frac{3\delta(p+1)}{4n(p+2)}}(t)J_2^{\frac{3}{2(p+2)}}(t)$, (41)

Where $\tilde{c}_1 = \frac{3\sqrt{2}}{\rho_0} > 0$, $\tilde{c}_2 = \frac{\sqrt{2}n(\rho_0+d)}{\rho_0}\cdot|\Omega|^{\frac{(2n-\delta)(p+1)}{2n(p+2)}} > 0$, (42) By Hölder inequality, we have $\int_\Omega u^{2n}dx \le \left(\int_\Omega u^{2n+r-1}dx\right)^{\frac{2n}{2n+r-1}} \cdot |\Omega|^{\frac{r-1}{2n+r-1}}$, (43)

from (43), we get $\int_\Omega u^{2n+r-1}dx \ge \varphi^{\frac{2n+r-1}{2n}} \cdot |\Omega|^{\frac{1-r}{2n}}$, (44)

Inserting (41) and (44) in (35), we obtain (45)as follows

$$\varphi'(t) \le c_1\varphi(t) + c_2\varphi^{\frac{3}{2}}(t) + \tilde{c}_3\varphi^{\frac{3\delta(p+1)}{4n(p+2)}}(t)J_2^{\frac{3}{2(p+2)}}(t) - 2n(2n-1)J_2(t) - 2nk_1\varphi^{\frac{2n+r-1}{2n}}(t)\cdot|\Omega|^{\frac{1-r}{2n}}$$

Next, we need to eliminate $J_2(t)$. By using Young inequality technique

$$\varphi^{\beta_1}(t)J_2^{\beta_2}(t) = (\gamma J_2(t))^{\beta_2}\left[\frac{\varphi^{\frac{\beta_1}{1-\beta_2}}(t)}{\gamma^{\frac{\beta_2}{1-\beta_2}}}\right]^{1-\beta_2} \le \gamma\beta_2 J_2(t) + (1-\beta_2)\gamma^{\frac{\beta_2}{\beta_2-1}}\varphi^{\frac{\beta_1}{1-\beta_2}}, \quad (46)$$

for $0<\beta_2<1$, where γ is a positive constant, then we have

$$\varphi^{\frac{3\delta(p+1)}{4n(p+2)}}(t)J_2^{\frac{3}{2(p+2)}}(t) \le \frac{3\gamma}{2(p+2)}J_2(t) + \frac{2p+1}{2(p+2)}\gamma^{-\frac{3}{2p+1}}\varphi^{\frac{3\delta(p+1)}{2n(2p+1)}}(t), \quad (47)$$

Substitute (47) in (45), and choose the position constant γ such that

$\tilde{c}_3\dfrac{3\gamma}{2(p+2)} - 2n(2n-1) = 0$, we obtain

$$\varphi'(t) \le c_1\varphi(t) + c_2\varphi^{\frac{3}{2}}(t) + c_3\varphi^{\frac{3\delta(p+1)}{2n(2p+1)}}(t) - 2nk_1\varphi^{\frac{2n+r-1}{2n}}(t)\cdot|\Omega|^{\frac{1-r}{2n}} \quad (48)$$

where $c_1 = \dfrac{nk_2M_1N}{\rho_0}$, $c_2 = \tilde{c}_1\left(\dfrac{nk_2M_1N}{\rho_0} + \dfrac{5n^2k_2M_1d}{2\mu\rho_0}\right)$, $c_3 = \tilde{c}_3\dfrac{2p+1}{2(p+2)}\gamma^{-\frac{3}{2p+1}}$,

$\tilde{c}_3 = \tilde{c}_2\left(\dfrac{nk_2M_1N}{\rho_0} + \dfrac{5n^2k_2M_1d}{2\mu\rho_0}\right)$. In the particular case $r = n+1$, the differential equality (48) can write to

$$\varphi'(t) \le c_1\varphi(t) + \bar{c}_2\varphi^{\frac{3}{2}}(t) + c_3\varphi^{\frac{3\delta(p+1)}{2n(2p+1)}}(t), \quad (49)$$

with $\bar{c}_2 = c_2 - 2nk_1|\Omega|^{-\frac{1}{2}}$. Integrating (91) over $[0,t]$, we obtain

$$t^* \ge \int_{\varphi(0)}^{\infty}\frac{1}{c_1\eta + \bar{c}_2\eta^{\frac{3}{2}} + c_3\eta^{\frac{3\delta(p+1)}{2n(2p+1)}}}.$$ This completes the proof.

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