AN ORIGINAL NOTE ON FOUR FAMOUS PRIMES PROBLEM

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ABSTRACT
In this paper, we use elementary complex calculus coupled with trivial arithmetic calculus and we show a simple Result on four famous primes problems (namely, the twin primes problem, the Fermat primes problem, the Mersenne primes problem and the Sophie Germain primes problem). This Result helps us to conjecture that the twin primes problem and the Fermat primes problem and the Mersenne primes problem and the Sophie Germain primes problem can be simultaneously solved by the same method.

Prime numbers are well known (see [1]). A *Fermat prime* (see [2]) is a prime number of the form $F_n = 2^{2^n} + 1$, where n is an integer ≥ 0. Fermat primes are characterized via divisibility in [2]. It is known (see [2]) that for every $j \in \{0,1,2,3,4\}$, F_j is a Fermat prime. The Fermat primes problem stipulates that there are infinitely many Fermat primes. An integer t is a *twin prime* (see [3]), if t is a prime ≥ 3 and if $t-2$ or $t+2$ is also a prime ≥ 3; for example, it is trivial to check that ($17,19$) and ($881,883$) and ($1019,1021$) are three couples of twin primes $\Big($ Primes and divisibility are well known, and for original characterizations of twin primes via divisibilty, see [3] $\Big)$. Twin primes are known for some integers >1021 and the twin primes problem asserts that there are infinitely many twin primes. A prime h is called a *Sophie Germain prime* (see [4] and [5]), if both h and $2h+1$ are prime; the first few Sophie Germain primes are $2,3,5,11,23,29,41,...$, and it is easy to check that 233 is a Sophie Germain prime. Sophie Germain primes are known for some integers >233 $\Big($ For original characterizations of Sophie Germain primes via divisibility, see [5] $\Big)$. The Sophie Germain primes problem asserts that there are infinitely many Sophie Germain primes. A *Mersenne prime* (see [4] or [5]) is a prime number of the form $M_m = 2^m - 1$, where m is prime. Primes and Mersenne primes are characterized via divisibility in [4] and [5]. It is known (see [4] or [5]) that M_2 and M_3 and M_{13} and M_{19} are Mersenne primes, and Mersenne primes are known for some integer $> M_{19}$. The Mersenne primes problem stipulates that there are infinitely many Mersenne primes. In this paper, we use elementary complex calculus coupled with trivial arithmetic calculus to show a simple Result on four famous primes problems (namely, the twin primes problem, the Fermat primes problem, the Mersenne primes problem and the Sophie Germain primes problem). This Result helps us to conjecture that the twin primes problem and the Fermat primes problem and the Mersenne primes problem and the Sophie Germain primes problem can be simultanously solved by the same method.

Keywords: *Fermat primes, twin primes, Mersenne primes, Sophie Germain primes, relative integers, tackle.*
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Preliminary. This paper is divided into two sections. In section.**1**, we introduce definitions that are not standard and we present some properties deduced from these definitions. In section.**2**, using elemenatary properties proved in section.**1** coupled with elementary complex calculus, elementary computation and trivial arithmetic calculus, we show an original result on four famous primes problems (namely, the twin primes problem, the Fermat primes problem, the Mersenne primes problem and the Sophie Germain primes problem). This Result helps us to conjecture that the twin primes problem and the Fermat primes problem and the Mersenne primes problem and the Sophie Germain primes problem can be simultaneously solved by the same method.

1. INTRODUCTION

In this section, we introduce definitions that are not standard and we present some elementary properties deduced from these definitions.

Definitions 1.0. For every integer $n \geq 2$, we define $H(n)$, h_n, $h_{n.1}$; $F(n)$, f_n, $f_{n.1}$; $M(n)$, m_n, $m_{n.1}$; $T(n)$, t_n and $t_{n.1}$ as follows:

$H(n) = \{x; 1 < x < 2n, x\, is\, a\, SophieGermain\, prime\}$, $h_n = \max_{h \in H(n)} h$, and

$$h_{n.1} = 2h_n^{h_n} \prod_{h \in H(n)} h$$

[observing (see Abstract and Definitions) that 2 and 3 and 5 are Sophie Germain primes, then it becomes immediate to deduce that for every integer $n \geq 3$, $\{2,3,5\} \subseteq H(n)$ and $h_n \geq 5$ and $h_{n.1} > 2 \times 5^5 \times 3 \times 5 > 93749$]; $F(n) = \{x; 1 < x < 2n, x\, is\, a\, Fermat\, prime\}$, $f_n = \max_{f \in F(n)} f$, and

$$f_{n.1} = 2 f_n^{f_n} \prod_{f \in F(n)} f$$

[observing (see Abstract and Definitions) that 3 and 5 are Fermat primes, then it becomes immediate to deduce that for every integer $n \geq 3$, $\{3,5\} \subseteq F(n)$ and $f_n \geq 5$ and $f_{n.1} \geq 2 \times 5^5 \times 3 \times 5 > 93749$];

$M(n) = \{x; 1 < x < 2n, x\, is\, a\, Mersenne prime\}$, $m_n = \max_{m \in M(n)} m$, and

$$m_{n.1} = 2m_n^{m_n} \prod_{m \in M(n)} m$$

[observing (see Abstract and Definitions) that 3 is a Mersenne prime, then it becomes immediate to deduce that for every integer $n \geq 3$, $3 \in M(n)$ and $m_n \geq 3$ and $m_{n.1} \geq 2 \times 3^3 \times 3 > 161$];

$T(n) = \{x; 1 < x < 2n, x\, is\, a\, twin\, prime\}$, $t_n = \max_{t \in T(n)} t$, and

$$t_{n.1} = 2t_n^{t_n} \prod_{t \in T(n)} t$$

[observing (see Abstract and Definitions) that 3 and 5 are twin primes, then it becomes immediate to deduce that for every integer $n \geq 3$, $\{3,5\} \subseteq T(n)$ and $t_n \geq 5$ and $t_{n.1} \geq 2 \times 5^5 \times 3 \times 5 > 93749$]. That being so, for every integer $n \geq 3$, consider ($h_{n.1}$, $f_{n.1}$, $m_{n.1}$, $t_{n.1}$) introduced above ; then $X(n.1)$ is defined as follows.

$$X(n.1) = (h_{n.1}, f_{n.1}, m_{n.1}, t_{n.1});$$

and we say that $x_{n.1} \in X(n.1)$ if $x_{n.1} = h_{n.1}$ or if $x_{n.1} = f_{n.1}$ or if $x_{n.1} = m_{n.1}$ or if $x_{n.1} = t_{n.1}$. Finally, we say that x is a *famous prime*, if x is a Fermat prime or a Mersenne prime or a Sophie Germain prime or a twin prime.

Using the previous definitions and denotations, let us remark.

Remark 1.1.*Let n be an integer ≥ 3 and let $X(n.1)$ introduced in Definitions 1.0. Now let $x_{n.1} \in X(n.1)$ and via $x_{n.1}$, look at x_n from which $x_{n.1}$ is attached. Then we have the following seven elementary properties.*

($1.1.0$.) x_n is a famous prime and $x_{n.1}$ is attached to x_n .

($1.1.1$.) If $x_{n.1} = f_{n.1}$, then $x_n = f_n$ and we are playing with a Fermat prime. If $x_{n.1} = h_{n.1}$, then $x_n = h_n$ and we are playing with a Sophie Germain prime. If $x_{n.1} = m_{n.1}$, then $x_n = m_n$ and we are playing with a Mersenne prime. If $x_{n.1} = t_{n.1}$, then $x_n = t_n$ and we are playing with a twin prime.

($1.1.2$.) $3 \leq x_n < x_{n.1}$; $x_{n.1}$ is even; $x_{n.1} > 2x_n^{1+x_n}$; $x_{n.1} > 161$; and

$$x_{n.1} = 2x_n^{x_n} \prod_{x \in U(n.1)} x,$$

$$(where U(n.1) = F(n) if\ x_{n.1} = f_{n.1}, U(n.1) = T(n) if\ x_{n.1} = t_{n.1},$$

$$U(n.1) = H(n) if\ x_{n.1} = h_{n.1}, and U(n.1) = M(n) if\ x_{n.1} = m_{n.1}).$$

(1.1.3.) **If** $x_n < n$, then $n > 3$ and $x_n = x_{n-1}$ and $x_{n.1} = x_{n-1.1}$.

(1.1.4.) **If** $x_{n.1} \leq 2n$, then $n > 4$ and $x_n < n$ and $x_{n.1} = x_{n-1.1}$.

(1.1.5) (The direct using of the famous prime x_n). **If** the famous prime x_n is of the form $x_n < n$, then $n > 3$ and $x_n = x_{n-1}$ and $x_{n.1} = x_{n-1.1}$.

(1.1.6) (The implicite using of the famous prime x_n). **If** $x_{n.1} \leq 2n$, then $n > 4$ and the famous prime x_n is of the form $x_n < n$, and $x_{n.1} = x_{n-1.1}$.

Proof. Property (1.1.0) and property (1.1.1) are immediate [**Indeed**, it suffices to use the definition of "a famous prime" and the defintion of $X(n.1)$ and the definition of $x_{n.1}$ (where $x_{n.1} \in X(n.1)$) and to observe that $x_{n.1}$ is attached to x_n]. Property (1.1.2) is trivial [**Indeed**, it suffices to use the definition of $(X(n.1), x_{n.1}, x_n)$ and the fact that 3 is a Fermat prime and 3 is a twin prime and 3 is a Sophie Germain prime and 3 is a Mersenne prime, and n is an integer ≥ 3 and $3 \in X(n.1)$]. Property (1.1.3) is trivial [**Indeed**, **if** $x_n < n$, clearly $n > 3$ (use the definition of x_n and observe that $x_n \geq 2n-3$ if $n = 3$, since 3 is a Fermat prime and 3 is a twin prime and 3 is a Sophie Germain prime and 3 is a Mersenne prime and n is an integer ≥ 3); consequently

$$x_n < n < 2n-2 \tag{1.1}$$

(1.1) immediately implies that

$$X(n.1) = X(n-1.1) \tag{1.2}$$

Using equality *(1.2)*, then we immediately deduce that

$$x_n = x_{n-1}\ and\ x_{n.1} = x_{n-1.1}.$$

Property (1.1.3) follows]. Property (1.1.4) is immediate [**Indeed**, if $x_{n.1} \leq 2n$, then using the previous inequality and the definition of x_n, it becomes trivial to deduce that

$$n > 4\ and\ x_n < n < 2n-2 \tag{1.3}$$

So $n > 4$ and $x_{n.1} = x_{n-1.1}$ (use *(1.3)* and property (1.1.3)). Property (1.1.4) follows]. Property (1.1.5) is the trivial reformulation of property (1.1.3) and property (1.1.6) is an immediate reformulation of property (1.1.4). Remark 1.1 follows.

Using the definition of ($h_{n.1}, t_{n.1}, f_{n.1}, m_{n.1}$) introduced in Definitions 1.0, then the following remark and proposition become immediate.

Remark 1.2. *We have the following two simple properties.*

(1.2.0). If $\lim_{n\to+\infty} h_{n.1} = +\infty$ and $\lim_{n\to+\infty} m_{n.1} = +\infty$ and $\lim_{n\to+\infty} t_{n.1} = +\infty$ and and $\lim_{n\to+\infty} f_{n.1} = +\infty$, then there are infinitely many Sophie Germain primes and there are infinitely many Mersenne primes and there are infinitely many twin primes and there are infinitely many Fermat primes.

(1.2.1). If for every integer $n \geq 3$, we have $h_{n.1} > n$ and $m_{n.1} > n$ and $f_{n.1} > n$ and $t_{n.1} > n$, then there are infinitely many Sophie Germain primes and there are infinitely many Mersenne primes and there are infinitely many Fermat primes and there are infinitely many twin primes.

Proof. Property (1.2.0) is immediate [Indeed, it suffices to use definitions of $h_{n.1}$ and $m_{n.1}$ and $t_{n.1}$ and $f_{n.1}$

(see Definitions 1.0)]. Property ($1.2.1$) is trivial [Clearly $\lim_{n\to+\infty} h_{n.1} = +\infty$ and $\lim_{n\to+\infty} m_{n.1} = +\infty$ and $\lim_{n\to+\infty} f_{n.1} = +\infty$ and $\lim_{n\to+\infty} t_{n.1} = +\infty$; therefore there are infinitely many Sophie Germain primes and there are infinitely many Mersenne primes and there are infinitely many Fermat primes and there are infinitely many twin primes (use the previous four equalities and apply property ($1.2.0$))].

Proposition 1.3. *If for every integer $n \geq 3$, and for every $x_{n.1} \in X(n.1)$ (see Definitions 1.0), we have $x_{n.1} > n$, then there are infinitely many Sophie Germain primes and and there are infinitely many Mersenne primes and there are infinitely many Fermat primes and there are infinitely many twin primes*.

Proof. Indeed, using the definition of $X(n.1)$, we immediately deduce that for every integer $n \geq 3$, we have $h_{n.1} > n$ and $m_{n.1} > n$ and $f_{n.1} > n$ and $t_{n.1} > n$; therefore, there are infinitely many Sophie Germain primes and there are infinitely many Mersenne primes and there are infinitely many Fermat primes and there are infinitely many twin primes [use the previous four inequalities and apply property ($1.2.1$) of Remark 1.2].

Proposition 1.3 clearly says that to prove simultaneously that there are infinitely many Sophie Germain primes and there are infinitely many Mersenne primes and there are infinitely many Fermat primes and there are infinitely many twin primes, it suffices to show that for every integer $n \geq 3$ and for every $x_{n.1} \in X(n.1)$, we have $x_{n.1} > n$. In this paper, we look at four famous primes problem (namely the twin primes problem, the Fermat primes problem, the Mersenne primes problem and the Sophie Germain primes problem) by showing the following original Result.

Result 1.4. *Let n be an integer ≥ 3 and look at $x_{n.1} \in X(n.1)$ (see Definition 1.0). Now let (ϕ_n, ν_n), where*

$$\phi_n = (ix_{n.1} - 4in - 4i + 1)^2 + 12x_{n.1} + x_{n.1}^2 + 12i + 4ix_{n.1} + 3ix_{n.1}^2, i^2 = -1;$$

and

$$\nu_n = 4(x_{n.1} - 2n - 2)^2 + (x_{n.1} - 2n)(1 + 2ix_{n.1} + 3ix_{n.1}^2), i^2 = -1.$$

Then

$x_{n.1} \geq 2n + 4$ or there exists a relative integer *k* such that

$$\phi_n + \nu_n = k(1 + 4x_{n.1} + 4i + 2ix_{n.1} + 3ix_{n.1}^2).$$

We recall that c' is a *relative integer* if c' is an integer ≥ 0 or if c' is an integer ≤ 0 (For example -8 and -3 and -1 and 0 and 71 and 241 are relative integers; $\frac{7}{4}$ is not a relative integer).

We will see in section.**2** that the previous Result helps us to conjecture that the twin primes problem and the Fermat primes problem and the Mersenne primes problem and the Sophie Germain primes problem can be simultaneously solved by the same method.

2. Elementary complex calculus and the proof of Result 1.4.

Recalls and Definitions 2.0 (*famous prime, Real numbers, complex numbers, relative integer,* Z, $Z(1 + 4x_{n.1} + 4i + 2ix_{n.1} + 3ix_{n.1}^2)$, *tackle of famous primes on* $1 + 4x_{n.1} + 4i + 2ix_{n.1} + 3ix_{n.1}^2$ and (ϕ_n, ν_n)).

We recall (see Definitions 1.0) that x is a *famous prime*, if x is a Fermat prime or a Mersenne prime or a Sophie Germain prime or a twin prime. It is known that R is the set all *real numbers* and θ is a *complex number* if $\theta = x + iy$, where $(x, y) \in R \times R$ and i is the complex entity satisfying $i^2 = -1$. We recall that c' is a *relative integer* if c' is an integer ≥ 0 or if c' is an integer ≤ 0 (For example -8 and -3 and -1 and 0 and 71 and 241 are relative integers; $\frac{7}{4}$ is not a relative integer). We recall that Z is the set of all relative integers; clearly

$$Z(1 + 4x_{n.1} + 4i + 2ix_{n.1} + 3ix_{n.1}^2) = \{k(1 + 4x_{n.1} + 4i + 2ix_{n.1} + 3ix_{n.1}^2); k \in Z\}.$$

That being so, we say that a function F is a *tackle of famous primes on* $1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2$, if the following properties (i) and (ii) are satisfied.

(i). For every integer $n\geq 3$ and for every $x_{n.1}\in X(n.1)$ (see Definition 1.0)

$$x_{n.1}\geq 2n+4 \, or \, F(x_{n.1})\in Z(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2).$$

(ii). For every integer $n\geq 3$ and for every $x_{n.1}\in X(n.1)$ such that $x_{n.1}\leq n$,

$$F(x_{n.1})\notin Z(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2).$$

(the previous definition gets sense, via the following: *Let* F' *be a function. If for every integer* $n\geq 3$ *and for every* $x_{n.1}\in X(n.1)$ *, we have* $x_{n.1}\geq 2n+4$, then F' is a *tackle of famous primes on* $1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2$. *Proof.* Properties (i) and (ii) mentioned above are clearly satisfied by F'.)

Now let n be integer ≥ 3 and let $x_{n.1}\in X(n.1)$ (Definitions 1.0). Then (ϕ_n,ν_n) is defined as follows.

$$\phi_n=(ix_{n.1}-4in-4i+1)^2+12x_{n.1}+x_{n.1}^2+12i+4ix_{n.1}+3ix_{n.1}^2, i^2=-1;$$

and

$$\nu_n=4(x_{n.1}-2n-2)^2+(x_{n.1}-2n)(1+2ix_{n.1}+3ix_{n.1}^2).$$

Now the following simple Theorem is our Result.

Theorem 2.1. *Let* n *be an integer* ≥ 3 *and let* $x_{n.1}\in X(n.1)$ *(see Definitions* 1.0*). Now look at* (ϕ_n,ν_n) *(see Recalls and Definitions* 2.0*). Then*

$$x_{n.1}\geq 2n+4 \, or \, \phi_n+\nu_n\in Z(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2),$$

where $Z(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2)$ is introduced in Recalls and Definitions 2.0.

Observe that Theorem 2.1 satisfies property (i) of the Definition of a *tackle of famous primes on* $1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2$ introduced in Definitions and Recalls 2.0. To prove Theorem 2.1, we need to use elementary complex calculus and trivial arithmetic calculus via the following proposition.

Proposition 2.2. *Let* n *be an integer* ≥ 3 *and let* $x_{n.1}\in X(n.1)$ *(see Definitions* 1.0*). Now look at* $(\phi_n,\nu_n,Z(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2))$ *(see Recalls and Definitions* 2.0*). If* $x_{n.1}\leq 2n$*, then we have the following two properties.*

($2.2.0$). $n>4$ and $x_{n.1}=x_{n-1.1}$.

($2.2.1$).

$$\phi_{n-1}+\nu_{n-1}-(\phi_n+\nu_n)\in Z(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2).$$

Proof. ($2.2.0$). Indeed, observing (by the hypotheses) that $x_{n.1}\leq 2n$, clearly $n>4$ and $x_{n.1}=x_{n-1.1}$ (use the previous inequality and property ($1.1.4$) of Remark 1.1). Property ($2.2.0$) follows.

($2.2.1$). Indeed, look at ϕ_n (see Recalls and Definitions 2.0) and consider ϕ_{n-1} (this consideration gets sense, since $n>4$ (use property ($2.2.0$))); then using the definition of ϕ_n, it becomes trivial to deduce that

$$\phi_{n-1}=(ix_{n-1.1}-4i(n-1)-4i+1)^2+12x_{n-1.1}+x_{n-1.1}^2+12i+4ix_{n-1.1}+3ix_{n-1.1}^2 \quad (2.1)$$

Since it is immediate that $(ix_{n-1.1}-4i(n-1)-4i+1)^2=(ix_{n-1.1}-4in-4i+1+4i)^2$, then using the previous equality, it becomes trivial to deduce that equality *(2.1)* says that

$$\phi_{n-1}=(ix_{n-1.1}-4in-4i+1+4i)^2+12x_{n-1.1}+x_{n-1.1}^2+12i+4ix_{n-1.1}+3ix_{n-1.1}^2 \quad (2.2)$$

Now look at *(2.2)*; noticing (by property $(2.2.0)$) that $x_{n.1}=x_{n-1.1}$ and using the previous equality, then it becomes trivial to deduce that equality *(2.2)* says that

$$\phi_{n-1} = (ix_{n.1} - 4in - 4i + 1 + 4i)^2 + 12x_{n.1} + x_{n.1}^2 + 12i + 4ix_{n.1} + 3ix_{n.1}^2 \tag{2.3}$$

Observing (by elementary computation) that

$$(ix_{n.1} - 4in - 4i + 1 + 4i)^2 = (ix_{n.1} - 4in - 4i + 1)^2 + 8i(ix_{n.1} - 4in - 4i + 1) - 16$$

and using the previous equality, it becomes trivial to deduce that equality *(2.3)* says that

$$\phi_{n-1} = (ix_{n.1} - 4in - 4i + 1)^2 + 12x_{n.1} + x_{n.1}^2 + 12i + 4ix_{n.1} + 3ix_{n.1}^2 + 8i(ix_{n.1} - 4in - 4i + 1) - 16 \tag{2.4}$$

Clearly

$$\phi_{n-1} = \phi_n + 8i(ix_{n.1} - 4in - 4i + 1) - 16 \tag{2.5}$$

(use the defintion of ϕ_n and *(2.4)*) . That being so, look at ν_n (see Recalls and Definitions 2.0) and consider ν_{n-1} (this consideration gets sense, since $n > 4$ (use property ($2.2.0$))); then using the definition of ν_n, it becomes trivial to deduce that

$$\nu_{n-1} = 4(x_{n-1.1} - 2(n-1) - 2)^2 + (x_{n-1.1} - 2(n-1))(1 + 2ix_{n-1.1} + 3ix_{n-1.1}^2) \tag{2.6}$$

Since it is immediate that $(x_{n-1.1} - 2(n-1) - 2)^2 = (x_{n-1.1} - 2n - 2 + 2)^2$ and $(x_{n-1.1} - 2(n-1)) = (x_{n-1.1} - 2n + 2)$, then using the previous two equalities, it becomes trivial to deduce that equality *(2.6)* says that

$$\nu_{n-1} = 4(x_{n-1.1} - 2n - 2 + 2)^2 + (x_{n-1.1} - 2n + 2)(1 + 2ix_{n-1.1} + 3ix_{n-1.1}^2) \tag{2.7}$$

Now look at *(2.7)*; noticing (by property $(2.2.0)$) that $x_{n.1} = x_{n-1.1}$ and using the previous equality, then it becomes trivial to deduce that equality *(2.7)* says that

$$\nu_{n-1} = 4(x_{n.1} - 2n - 2 + 2)^2 + (x_{n.1} - 2n + 2)(1 + 2ix_{n.1} + 3ix_{n.1}^2) \tag{2.8}$$

Observing (by elementary computation) that

$$(x_{n.1} - 2n - 2 + 2)^2 = (x_{n.1} - 2n - 2)^2 + 4(x_{n.1} - 2n - 2) + 4$$

and

$$(x_{n.1} - 2n + 2)(1 + 2ix_{n.1} + 3ix_{n.1}^2) = (x_{n.1} - 2n)(1 + 2ix_{n.1} + 3ix_{n.1}^2) + 2(1 + 2ix_{n.1} + 3ix_{n.1}^2),$$

then using the previous two equalities, it becomes trivial to deduce that equality *(2.8)* says that

$$\nu_{n-1} = 4(x_{n.1} - 2n - 2)^2 + (x_{n.1} - 2n)(1 + 2ix_{n.1} + 3ix_{n.1}^2) + \lambda_n \tag{2.9}$$

where

$$\lambda_n = 16(x_{n.1} - 2n - 2) + 16 + 2(1 + 2ix_{n.1} + 3ix_{n.1}^2) \tag{2.9'}$$

Clearly

$$\nu_{n-1} = \nu_n + 16(x_{n.1} - 2n - 2) + 16 + 2(1 + 2ix_{n.1} + 3ix_{n.1}^2) \tag{2.10}$$

(use the defintion of ν_n and *(2.9)* and $(2.9')$) . Using equalities *(2.5)* and *(2.10)*, then we immediatly deduce that

$$\phi_{n-1} + \nu_{n-1} = \phi_n + \nu_n + 8i(ix_{n.1} - 4in - 4i + 1) - 16 + 16(x_{n.1} - 2n - 2) + 16 + 2(1 + 2ix_{n.1} + 3ix_{n.1}^2) \tag{2.11}$$

Equality *(2.11)* clearly says that

$$\phi_{n-1} + \nu_{n-1} - (\phi_n + \nu_n) = 8i(ix_{n.1} - 4in - 4i + 1) - 16 + 16(x_{n.1} - 2n - 2) + 16 + 2(1 + 2ix_{n.1} + 3ix_{n.1}^2) \tag{2.12}$$

Now look at equality *(2.12)*; it is very easy to check (by elementary computation and the fact that $i^2 = -1$) that

$$8i(ix_{n.1} - 4in - 4i + 1) - 16 + 16(x_{n.1} - 2n - 2) + 16 + 2(1 + 2ix_{n.1} + 3ix_{n.1}^2) = 2(1 + 4x_{n.1} + 4i + 2ix_{n.1} + 3ix_{n.1}^2).$$

Using the previous equality and equality *(2.12)*, then we immediately deduce that

$$\phi_{n-1} + \nu_{n-1} - (\phi_n + \nu_n) = 2(1 + 4x_{n.1} + 4i + 2ix_{n.1} + 3ix_{n.1}^2) \tag{2.13}$$

Clearly

$$\phi_{n-1} + \nu_{n-1} - (\phi_n + \nu_n) \in Z(1 + 4x_{n.1} + 4i + 2ix_{n.1} + 3ix_{n.1}^2)$$

(use (2.13) and the definition of $Z(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2)$) . Property ($2.2.1$) follows and Proposition 2.2 immediately follows.
This simple Proposition proved, we are ready to give an elementary proof of Theoreme 2.1.

Proof of Theorem 2.1. Otherwise (we reason by reduction to absurd), let n be a minimum counter-example to Theorem 2.1; then

$$x_{n.1} < 2n+4 \, and \, \phi_n + \nu_n \notin Z(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2) \tag{2.14}$$

and we observe the following.
Observation. $2.1i$. *Look at* n *(recall* n *is a minimum counter-example to Theorem* 2.1*), and let* $x_{n.1}$*. Then* $x_{n.1} \leq 2n$.
Otherwise,

$$x_{n.1} > 2n. \tag{2.15}$$

Observing that $x_{n.1}$ is even (use the definition of $x_{n.1}$) and since $2n$ is trivially even, then using the previous, it becomes immediate to deduce that inequality *(2.15)* implies that

$$x_{n.1} \geq 2n+2 \tag{2.16}$$

That being so, remarking (by *(2.14)*) that

$$x_{n.1} < 2n+4 \tag{2.17}$$

and noticing that $(x_{n.1}, 2n+4)$ is a couple of even numbers ($x_{n.1}$ is even (use the definition of $x_{n.1}$) and $2n+4$ is trivially even), then using the previous, it becomes immediate to deduce that inequality *(2.17)* implies that

$$x_{n.1} \leq 2n+2. \tag{2.18}$$

So

$$x_{n.1} = 2n+2 \tag{2.19}$$

(use *(2.16)* and *(2.18)*). Now let ϕ_n (see Recalls and Definitions 2.0) and observe

$$\phi_n = (ix_{n.1} - 4in - 4i + 1)^2 + 12x_{n.1} + x_{n.1}^2 + 12i + 4ix_{n.1} + 3ix_{n.1}^2 \tag{2.20}$$

Observing (by *(2.19)*) that $x_{n.1} = 2n+2$, clearly $2n = x_{n.1} - 2$; now replacing intelligently $2n$ by $x_{n.1} - 2$, then it becomes immediate to deduce that equality *(2.20)* is of the form

$$\phi_n = (ix_{n.1} - 2ix_{n.1} + 1)^2 + 12x_{n.1} + x_{n.1}^2 + 12i + 4ix_{n.1} + 3ix_{n.1}^2 \tag{2.21}$$

It is trivial to deduce that equality *(2.21)* says that

$$\phi_n = (-ix_{n.1} + 1)^2 + 12x_{n.1} + x_{n.1}^2 + 12i + 4ix_{n.1} + 3ix_{n.1}^2 \tag{2.22}$$

It is easy to check (by elementary computation and the fact that $i^2 = -1$) that equality *(2.22)* says that

$$\phi_n = 1 + 12x_{n.1} + 12i + 2ix_{n.1} + 3ix_{n.1}^2 \tag{2.23}$$

That being so, let ν_n (see Recalls and Definitions 2.0) and observe that

$$\nu_n = 4(x_{n.1} - 2n - 2)^2 + (x_{n.1} - 2n)(1 + 2ix_{n.1} + 3ix_{n.1}^2) \tag{2.24}$$

Observing (by *(2.19)*) that $x_{n.1} = 2n+2$, clearly $2n = x_{n.1} - 2$; now replacing intelligently $2n$ by $x_{n.1} - 2$, then it becomes immediate to deduce that equality *(2.24)* is of the form

$$\nu_n = 4(x_{n.1} - x_{n.1})^2 + (x_{n.1} - x_{n.1} + 2)(1 + 2ix_{n.1} + 3ix_{n.1}^2) \tag{2.25}$$

It is trivial to deduce that equality *(2.25)* says that

$$\nu_n = 2(1 + 2ix_{n.1} + 3ix_{n.1}^2) \tag{2.26}$$

Using equalities *(2.23)* and *(2.26)*, then we immediately deduce that

$$\phi_n + \nu_n = 3(1 + 4x_{n.1} + 4i + 2ix_{n.1} + 3ix_{n.1}^2) \tag{2.26'}$$

Clearly

$$(\phi_n + \nu_n) \in Z(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2) \quad (2.27)$$

(use $(2.26')$ and the definition of $Z(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2)$) . *(2.27)* contradicts *(2.14)*. Observation. $2.1i$ follows.

Observation. $2.1ii$. *Look at* n. *Then* $n>4$ *and* $x_{n.1}=x_{n-1.1}$.

This Observation is immediate, by using Observation. $2.1i$ and Property $2.2.0$ of Proposition 2.2.

Observation. $2.1iii$. *Look at* n, *and consider* (ϕ_{n-1},ν_{n-1}) *(this consideration gets sense, since* $n>4$ *(by using Observation.* $2.1ii$ *)). Then*

$$\phi_{n-1}+\nu_{n-1}-(\phi_n+\nu_n) \in Z(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2).$$

Indeed observe (by Observation. $2.1i$ and Observation. $2.1ii$) that

$$x_{n.1} \le 2n \, and \, n>4 \quad (2.28)$$

Now using *(2.28)*, it becomes trivial to deduce that all the hypotheses of Proposition 2.2 are satisfied, therefore, all the conclusions of Proposition 2.2 are satisfied; in particular property ($2.2.1$) of Proposition 2.2 is satisfied; consequently

$$\phi_{n-1}+\nu_{n-1}-(\phi_n+\nu_n) \in Z(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2).$$

Observation. $2.1iii$ follows.

Observation. $2.1iv$. *Look at* n *(recall* n *is a minimum counter-example to Theorem* 2.1*), and consider* (ϕ_{n-1},ν_{n-1}) *(this consideration gets sense, since* $n>4$ *(by using Observation.* $2.1ii$ *)). Then*

$$\phi_{n-1}+\nu_{n-1} \in Z(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2).$$

Indeed look at n (recall n is a minimum counter-example to Theorem 2.1), and via n, consider $n-1$ (this consideration gets sense, since $n>4$ (by using Observation. $2.1ii$)); then, by the minimality of n, $n-1$ is not a counter-example to Theorem 2.1; so

$$x_{n-1.1} \ge 2(n-1)+4 \, or \, \phi_{n-1}+\nu_{n-1} \in Z(1+4x_{n-1.1}+4i+2ix_{n-1.1}+3ix_{n-1.1}^2) \quad (2.29)$$

Observing (by Observation. $2.1ii$) that $x_{n.1}=x_{n-1.1}$ and using the previous equality, then it becomes immediate to deduce that *(2.29)* clearly says that

$$x_{n.1} \ge 2n+2 \, or \, \phi_{n-1}+\nu_{n-1} \in Z(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2) \quad (2.29')$$

Clearly

$$\phi_{n-1}+\nu_{n-1} \in Z(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2)$$

(use $(2.29')$ and Observation. $2.1i$). Observation. $2.1iv$ follows.

Observation. $2.1.v$. *Look at* n *and let* (ϕ_n,ν_n). *Then*

$$\phi_n+\nu_n \in Z(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2).$$

Indeed let

$$x \in Z \, such that \, \phi_{n-1}+\nu_{n-1}-(\phi_n+\nu_n) = x(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2) \quad (2.30)$$

such a x exists (use the definition of $Z(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2)$ and Observation. $2.1iii$). Now

$$let \, x' \in Z \, such \, that \, \phi_{n-1}+\nu_{n-1} = x'(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2) \quad (2.31)$$

such a x' exists (use the definition of $Z(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2)$ and Observation. $2.1iv$). That being so, using equality of *(2.31)*, then it becomes trivial to deduce that equality of *(2.30)* clearly says that

$$x'(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2)-(\phi_n+\nu_n) = x(1+4x_{n.1}+4i+2ix_{n.1}+3ix_{n.1}^2) \quad (2.32)$$

It is trivial to deduce that equality *(2.32)* clearly says that

$$(\phi_n + \nu_n) = (x' - x)(1 + 4x_{n.1} + 4i + 2ix_{n.1} + 3ix_{n.1}^2) \tag{2.33}$$

Clearly $\phi_n + \nu_n \in Z(1 + 4x_{n.1} + 4i + 2ix_{n.1} + 3ix_{n.1}^2)$ (use equality *(2.33)* and observe that $x' - x \in Z$). Observation. $2.1.\nu$ follows.

These simple observations made, then it becomes trivial to see that Observation. $2.1.\nu$ clearly contradicts *(2.14)*. Theorem 2.1 follows.

Now using the definition of a *tackle of famous primes on* $1 + 4x_{n.1} + 4i + 2ix_{n.1} + 3ix_{n.1}^2$ (see Recalls and Definitions 2.0), then Theorem 2.1 implies that the twin primes problem and the Fermat primes problem and the Mersenne primes problem and the Sophie Germain primes problem can be simultanously solved by the same method, via the following remark.

Remark 2.3. *If there exists a function* F *which is a tackle of famous primes on* $1 + 4x_{n.1} + 4i + 2ix_{n.1} + 3ix_{n.1}^2$, *then there are infinitely many Sophie Germain primes and there are infinitely many Mersenne primes and there are infinitely many Fermat primes and there are infinitely many twin primes.*

Proof.

$$\text{Let } n \text{ be an integer} \geq 3 \text{ and let } x_{n.1} \in X(n.1)\text{; then } x_{n.1} > n \tag{2.34}$$

(otherwise $x_{n.1} \leq n$ and the previous inequality contradicts property (ii) of Definition of a *tackle of famous primes on* $1 + 4x_{n.1} + 4i + 2ix_{n.1} + 3ix_{n.1}^2$ introduced in Definitions and Recalls 2.0, since F is supposed to be a tackle of famous primes on $1 + 4x_{n.1} + 4i + 2ix_{n.1} + 3ix_{n.1}^2$) . Using *(2.34)* and Proposition 1.3, then we immediately deduce that there are infinitely Mersenne primes and there are infinitely many Sophie Germain primes and there are infinitely many Fermat primes and there are infinitely many twin primes.

It is immediate that the definition of a *tackle of famous primes on* $1 + 4x_{n.1} + 4i + 2ix_{n.1} + 3ix_{n.1}^2$, has a little resemblance with Theorem 2.1 (more precisely, Theorem 2.1 satisfies property (i) of the Definition of a *tackle of famous primes on* $1 + 4x_{n.1} + 4i + 2ix_{n.1} + 3ix_{n.1}^2$ introduced in Definitions and Recalls 2.0). Now using Remark 2.3, then it becomes natural and not surprising to conjecture the following.

Conjecture 2.4. *There exists a function* F *which is a tackle of famous primes on* $1 + 4x_{n.1} + 4i + 2ix_{n.1} + 3ix_{n.1}^2$

Via Remark 2.3, then it becomes immediate to see that Conjecture 2.4 simultaneously implies the Fermat primes problem and twin primes problem and Sophie Germain primes problem and Mersenne primes problem.

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