# BLOW-UP FOR SOLUTIONS TO A KIND OF QUASI-LINEAR SYSTEMS

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**ABSTRACT**

The focus of this paper is on the geometric mechanism for blow-up of solutions to Cauchy problem for a kind of quasi-linear systems. We demonstrate that blow-up is closely related to the eigenvalues of matrix by using Cramer criterion and characteristics method.

**Keywords**: *blow-up, characteristics, conservation laws.*
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## 1. INTRODUCTION

S. Alinahc has made systematic study on the blow-up phenomena for quasi-linear equation or systems (see [1, 2, 3, 4, 5, 6]). In [5] he proposes two local blow-up mechanisms: one is the Ordinary Differential equations blow-up" mechanism which occurs mainly in ordinary differential equations or semi-linear equations or systems, the other is Geometric blow-up" mechanism which occurs mainly in quasi-linear equations or systems. By introducing the concept of blow-up system, S. Alinahc gives some abstract-looking blow-up results for general quasi-linear systems. In [7], M. Grassin proves that the existence of global smooth solutions to Euler equations for a perfect gas is related to the eigenvalues of the gradient matrix of initial datum. We are interested in the precise relationship between the eigenvalues and the blow-up occurrence and get some concrete results by choosing special quasi-linear equations or systems.

In the remaining part, we first study the blow-up phenomena for equations, then for systems.

## 2 Blow-up for multidimensional quasi-linear equations or systems

From the results of S. Alinahc, we know that it is usually the partial derivatives of solutions that blow up for quasi-linear equations or systems, and local solutions exist in most situations, so we always assume that the solutions exist locally, and only consider the blow-up of first-order partial derivatives of solutions. We give some notations:

$x=(x_1,\cdots,x_n)\in\mathbb{R}^n, u\in\mathbb{R}^1, x_0=(x_1^0,\cdots,x_n^0), U=(u_1,\cdots,u_m)\in\mathbb{R}^m, U_0=(u_1^0,\cdots,u_m^0), n\ge 1, m\ge 1, I$ is the unit matrix.

### 2.1 Blow-up for scalar equations

First we study the following Cauchy problemï¿½ï¿½

$$\begin{cases} u_t+\sum_{i=1}^{n} u\partial_{x_i}u=0,\ x\in\mathbb{R}^n,\ t>0, \\ u(x,t)|_{t=0}=u_0(x),\ x\in\mathbb{R}^n. \end{cases} \tag{1}$$

The characteristic satisfies the following ODE:

$$\begin{cases} dx_i dt=u(x(t),t),\ i=1,2,\cdots,n, \\ x(0)=x_0. \end{cases} \tag{2}$$

Combining (1) with (2), we have $dudt(x(t),t)=0$, so $u(x(t),t)=u(x_0,0)=u_0(x_0)$, by equation (2) we have $x_i(t)=x_i^0+tu_0(x_0), i=1,2,\cdots,n$.

Let

$$A^T=\partial x\partial x_0=\partial(x_1,\cdots,x_n)\partial(x_1^0,\cdots,x_n^0),$$
$$b^T=\partial u\partial x=(\partial_{x_1}u,\cdots,\partial_{x_n}u),$$

$$c^T = \partial u_0 \partial x_0 = (\partial_{x_1^0} u_0, \cdots, \partial_{x_n^0} u_0),$$

then $Ab = c$ and

$$A^T = I + t\partial(u_0, \cdots, u_0)\partial(x_1^0, \cdots, x_n^0) \overset{\Delta}{=} (a_1, \cdots, a_n)^T.$$

**Theorem 2.1** *If* $\sum_{i=1}^{n} \partial_{x_i^0} u_0 \geq 0, \forall x_0 \in \mathbb{R}^n$, *then the first-order partial derivatives of solutions to equation (1) do not blow up. If there exists some* $x_0 \in \mathbb{R}^n$ *satisfying* $\sum_{i=1}^{n} \partial_{x_i^0} u_0 < 0$, *denoting* $t_1 = -(\sum_{i=1}^{n} \partial_{x_i^0} u_0)^{-1}$, *and if there also exists some* $i_0$, $1 \leq i_0 \leq n$ *such that*

$$\det(a_1, \cdots, a_{i_0-1}, c, a_{i_0+1}, \cdots, a_n)|_{t=t_1} \neq 0,$$

then $\lim_{t \to t_1} |\partial_{x_{i_0}} u| = \infty.$

**Proof** Since $Rank(\partial(u_0, \cdots, u_0)\partial(x_1^0, \cdots, x_n^0)) = 1$, from the basic theory of matrix, we know that the eigenvalues of $\partial(u_0, \cdots, u_0)\partial(x_1^0, \cdots, x_n^0)$ are $\overbrace{0, \cdots, 0}^{n-1}$, $\sum_{i=1}^{n} \partial_{x_i^0} u_0$. So the eigenvalues of $A$ are $\overbrace{1, \cdots, 1}^{n-1}$, $1 + t\sum_{i=1}^{n} \partial_{x_i^0} u_0$. If $\sum_{i=1}^{n} \partial_{x_i^0} u_0 \geq 0$, then $\det A = 1 + t\sum_{i=1}^{n} \partial_{x_i^0} u_0 > 0$, i.e., $A^{-1}$ exist for all $t \in \mathbb{R}$. Thus the first-order partial derivatives do not blow up. If $\sum_{i=1}^{n} \partial_{x_i^0} u_0 < 0$, then when $t = t_1 = -(\sum_{i=1}^{n} \partial_{x_i^0} u_0)^{-1}$, $\det A = 1 + t_1 \sum_{i=1}^{n} \partial_{x_i^0} u_0 = 0$. Since

$$\det(a_1, \cdots, a_{i_0-1}, c, a_{i_0+1}, \cdots, a_n) \neq 0,$$

by the Crame criterion, we have that

$$\lim_{t \to t_1} |\partial_{x_{i_0}} u| = \left| \det(a_1, \cdots, a_{i_0-1}, c, a_{i_0+1}, \cdots, a_n) \det(a_1, \cdots, \cdots, a_n) \right|_{t=t_1} = \infty.$$

So far we complete the proof of the theorem.

Next we consider the equation with source term:

$$\begin{cases} u_t + \sum_{i=1}^{n} u \partial_{x_i} u = -\alpha u, \; x \in \mathbb{R}^n, \; t > 0, \\ u(x,t)|_{t=0} = u_0(x), \; x \in \mathbb{R}^n, \end{cases} \tag{3}$$

$\alpha \neq 0$ is a constant. By using equation (2) we get $dudt(x(t),t) = -\alpha u$, so $u(x(t),t) = u_0(x_0)e^{-\alpha t}$, then $x_i(t) = x_i^0 + 1 - e^{-\alpha t} \alpha u_0(x_0) =: x_i^0 + \beta(t) u_0(x_0), i = 1, 2, \cdots, n$, where $\beta(t) = 1 - e^{-\alpha t} \alpha > 0.$ Let

$$A^T = \partial x \partial x_0 = \partial(x_1, \cdots, x_n)\partial(x_1^0, \cdots, x_n^0),$$

$$b^T = \partial u \partial x = (\partial_{x_1} u, \cdots, \partial_{x_n} u),$$

$$c^T = \partial u_0 \partial x_0 = (\partial_{x_1^0} u_0, \cdots, \partial_{x_n^0} u_0),$$

we have

$$A^T = I + \beta(t)\partial(u_0, \cdots, u_0)\partial(x_1^0, \cdots, x_n^0) \overset{\Delta}{=} (a_1, \cdots, a_n)^T, \; Ab = ce^{-\alpha t}.$$

Correspondingly we have the theorem:

**Theorem 2.2** *If* $\forall x_0 \in \mathbb{R}^n$, $\sum_{i=1}^n \partial_{x_i^0} u_0 \geq -\alpha (\alpha > 0)$, *or* $\sum_{i=1}^n \partial_{x_i^0} u_0 \geq 0 (\alpha < 0)$ *holds, then the first-order partial derivatives of solutions to equation (3) do not blow up. If there exists some* $x_0 \in \mathbb{R}^n$ *satisfying* $\sum_{i=1}^n \partial_{x_i^0} u_0 < -\alpha (\alpha > 0)$, *or* $\sum_{i=1}^n \partial_{x_i^0} u_0 < 0 (\alpha < 0)$, *denoting* $t_1 = -1\alpha \ln(1 + \alpha \sum_{i=1}^n \partial_{x_i^0} u_0)$, *and if there also exists some* $i_0$, $1 \leq i_0 \leq n$ *such that*

$$\det(a_1, \cdots, a_{i_0 - 1}, c, a_{i_0 + 1}, \cdots, a_n)|_{t=t_1} \neq 0,$$

then $\lim_{t \to t_1} |\partial_{x_{i_0}} u| = \infty.$

**Proof** Since $Rank(\partial(u_0, \cdots, u_0)\partial(x_1^0, \cdots, x_n^0)) = 1$, From Lemma 2.1, we know that the eigenvalues of $\partial(u_0, \cdots, u_0)\partial(x_1^0, \cdots, x_n^0)$ are $\overbrace{0, \cdots, 0}^{n-1}$, $\sum_{i=1}^n \partial_{x_i^0} u_0$, so the eigenvalues of $A$ are $\overbrace{1, \cdots, 1}^{n-1}$, $1 + \beta(t)\sum_{i=1}^n \partial_{x_i^0} u_0$. If $\sum_{i=1}^n \partial_{x_i^0} u_0 \geq -\alpha (\alpha > 0)$, or $\sum_{i=1}^n \partial_{x_i^0} u_0 \geq 0 (\alpha < 0)$ holds, then $\det A = 1 + \beta(t)\sum_{i=1}^n \partial_{x_i^0} u_0 > 0$, here $b = A^{-1} ce^{-\alpha t}$, so the first-order partial derivatives do not blow up. If $\sum_{i=1}^n \partial_{x_i^0} u_0 < -\alpha (\alpha > 0)$, or $\sum_{i=1}^n \partial_{x_i^0} u_0 < 0 (\alpha < 0)$ holds, $\det A|_{t=t_1} = 1 + \beta(t_1)\sum_{i=1}^n \partial_{x_i^0} u_0 = 0$, since $\det(a_1, \cdots, a_{i_0 - 1}, c, a_{i_0 + 1}, \cdots, a_n) \neq 0$, by Cramer criterion we have

$$\lim_{t \to t_1} |\partial_{x_{i_0}} u| = \left| \det(a_1, \cdots, a_{i_0 - 1}, ce^{-\alpha t}, a_{i_0 + 1}, \cdots, a_n) \det(a_1, \cdots, \cdots, a_n) \right|_{t=t_1} = \infty.$$

So far the proof of the theorem is completed.

Consider the following Cauchy problem of more general scalar equation:

$$\begin{cases} u_t + \sum_{i=1}^n f_i(u)\partial_{x_i} u = 0, \; x \in \mathbb{R}^n, \; t > 0, \\ u(x,t)|_{t=0} = u_0(x), \; x \in \mathbb{R}^n, \end{cases} \tag{4}$$

where $f_i \in C^\infty(\mathbb{R}^1)$, $i = 1, \cdots, n.$ By considering the following ordinary differential system

$$\begin{cases} dx_i dt = f_i(u(x(t),t)), \; i = 1,2,\cdots,n, \\ x(0) = x_0, \end{cases}$$

we get $dudt(x(t),t)=0$, so $u(x(t),t)=u(x_0,0)=u_0(x_0)$, moreover $x_i(t)=x_i^0+tf_i(u_0(x_0)), i=1,2,\cdots,n$. Let

$$A^T=\partial x\partial x_0=\partial(x_1,\cdots,x_n)\partial(x_1^0,\cdots,x_n^0),$$

$$b^T=\partial u\partial x=(\partial_{x_1}u,\cdots,\partial_{x_n}u),$$

$$c^T=\partial u_0\partial x_0=(\partial_{x_1^0}u_0,\cdots,\partial_{x_n^0}u_0),$$

we have

$$A^T=I+t\partial(f_1(u_0(x_0)),\cdots,f_n(u_0(x_0)))\partial(x_1^0,\cdots,x_n^0)\overset{\Delta}{=}(a_1,\cdots,a_n)^T,\ Ab=c$$

Similar to the proof of Theorem 2.2, we get the following theorem:

**Theorem 2.3** *If* $\sum_{i=1}^{n}\partial_{x_i^0}f_i(u_0(x_0))\geq 0, \forall x_0\in\mathbb{R}^n$, *then the first-order partial derivatives of solutions to equation (4) do not blow up. If there exists some* $x_0\in\mathbb{R}^n$ *satisfying* $\sum_{i=1}^{n}\partial_{x_i^0}f_i(u_0(x_0))<0$, *denoting* $t_1=-(\sum_{i=1}^{n}\partial_{x_i^0}f_i(u_0(x_0)))^{-1}$, *and if there also exists some* $i_0,\ 1\leq i_0\leq n$ *such that*

$$\det(a_1,\cdots,a_{i_0-1},c,a_{i_0+1},\cdots,a_n)|_{t=t_1}\neq 0,$$

then $\lim_{t\to t_1}|\partial_{x_{i_0}}u|=\infty$.

For the following equation:

$$\begin{cases} u_t+\sum_{i=1}^{n}f_i(u)\partial_{x_i}u=-\alpha u,\ x\in\mathbb{R}^n,\ t>0, \\ u(x,t)|_{t=0}=u_0(x),\ x\in\mathbb{R}^n, \end{cases} \tag{5}$$

where $\alpha\neq 0$ is a constant, $f_i\in C^\infty(\mathbb{R}^1), i=1,\cdots,n.$ Consider the ordinary differential system,

$$\begin{cases} dx_idt=f_i(u(x(t),t)), i=1,2,\cdots,n, \\ x(0)=x_0. \end{cases}$$

We get that $u(x(t),t)=u_0(x_0)e^{-\alpha t}$, it yields that

$$x_i(t)=x_i^0+\int_0^t f_i(u_0(x_0)e^{-\alpha\tau})d\tau, i=1,\cdots,n.$$

Let

$$A^T=\partial x\partial x_0=\partial(x_1,\cdots,x_n)\partial(x_1^0,\cdots,x_n^0),$$

$$b^T=\partial u\partial x=(\partial_{x_1}u,\cdots,\partial_{x_n}u),$$

$$c^T=\partial u_0\partial x_0=(\partial_{x_1^0}u_0,\cdots,\partial_{x_n^0}u_0),$$

then

$$A^T = I + \begin{pmatrix} \beta_1(t) \\ \vdots \\ \beta_n(t) \end{pmatrix} (\partial_{x_1^0} u_0, \cdots, \partial_{x_n^0} u_0) \overset{\Delta}{=} (a_1, \cdots, a_n)^T,$$

$$Ab = ce^{-\alpha t},$$

where $\beta_i(t) = f_i(u_0(x_0)) - f_i(u_0(x_0)e^{-\alpha t})\alpha u_0(x_0)$. By the similar computation, the eigenvalues of $A$ are $\overbrace{1, \cdots, 1}^{n-1}$, $1 + \sum_{i=1}^{n} \beta_i(t) \partial_{x_i^0} u_0$. Correspondingly, we have

**Theorem 2.4** *Let* $\beta_i(t) = f_i(u_0(x_0)) - f_i(u_0(x_0)e^{-\alpha t})\alpha u_0(x_0)$. *If* $1 + \sum_{i=1}^{n} \beta_i(t) \partial_{x_i^0} u_0 \neq 0, \forall t > 0, \forall x_0 \in \mathbb{R}^n$, *then the first-order partial derivatives of solutions to equation (5) do not blow up. If there exist some* $t_1 > 0$ *and some* $x_0 \in \mathbb{R}^n$ *such that* $1 + \sum_{i=1}^{n} \beta_i(t_1) \partial_{x_i^0} u_0 = 0$, *and if there also exists some* $i_0$, $1 \leq i_0 \leq n$ *such that* $\det(a_1, \cdots, a_{i_0-1}, ce^{-\alpha t}, a_{i_0+1}, \cdots, a_n)|_{t=t_1} \neq 0$, *then* $\lim_{t \to t_1} |\partial_{x_{i_0}} u| = \infty$.

### 2.2 Blow-up for systems

First we study the following system:

$$\begin{cases} U_t + \sum_{i=1}^{n} f_i(U) \partial_{x_i} U = 0, \; x \in \mathbb{R}^n, \; t > 0, \\ U(x,t)|_{t=0} = U_0(x), \; x \in \mathbb{R}^n, \end{cases} \tag{6}$$

where $f_i \in C^\infty(\mathbb{R}^m)$, $i = 1, \cdots, n$. Consider the following ordinary differential system:

$$\begin{cases} dx_i dt = f_i(U(x(t), t)), \; i = 1, 2, \cdots, n, \\ x(0) = x_0, \end{cases}$$

then $dUdt(x(t), t) = 0$, so $U(x(t), t) = U(x_0, 0) = U_0(x_0)$, it yields that $x_i(t) = x_i^0 + t f_i(U_0(x_0)), i = 1, \cdots, n$.

Let

$$A^T = \partial x \partial x_0 = \partial(x_1, \cdots, x_n) \partial(x_1^0, \cdots, x_n^0),$$

$$b_j^T = \partial u_j \partial x = (\partial_{x_1} u_j, \cdots, \partial_{x_n} u_j),$$

$$c_j^T = \partial u_j^0 \partial x_0 = (\partial_{x_1^0} u_j^0, \cdots, \partial_{x_n^0} u_j^0), \; j = 1, 2, \cdots, m.$$

then

$$A^T = I + t\partial(f_1(U_0(x_0)), \cdots, f_n(U_0(x_0)))\partial(u_1^0, \cdots, u_m^0) \cdot \partial(u_1^0, \cdots, u_m^0)\partial(x_1^0, \cdots, x_n^0) = (a_1, \cdots, a_n)^T,$$

$$Ab_j = c_j,\ j = 1,\cdots,m.$$

Denote

$$D = \partial(f_1(U_0(x_0)),\cdots,f_n(U_0(x_0)))\partial(u_1^0,\cdots,u_m^0)\cdot\partial(u_1^0,\cdots,u_m^0)\partial(x_1^0,\cdots,x_n^0),$$

then $A^T = I + tD.$ Assume that the eigenvalues of $D$ are $\{\lambda_i(x_0), i = 1,\cdots,n\}$, then the eigenvalues of $A$ are $\{1+t\lambda_i(x_0), i = 1,\cdots,n\}$. If $\lambda_i(x_0) > 0$, $1 \le i \le n, \forall x_0 \in \mathbb{R}^n$, then $\det A = \prod_{i=1}^{n}(1+t\lambda_i(x_0)) > 0$, here we have $b_j = A^{-1}c_j,\ j = 1,\cdots,m.$ So the first-order partial derivatives of solutions do not blow up. If there exist some $i_0$, $1 \le i_0 \le n$ and some $x_0 \in \mathbb{R}^n$, such that $\lambda_{i_0}(x_0) < 0$, denoting $t_1 = -(\lambda_{i_0}(x_0))^{-1}$, then $\det A|_{t=t_1} = 0.$ If there also exist some $j_0, j_1$, $1 \le j_0 \le n, 1 \le j_1 \le m$, satisfying $\det(a_1,\cdots,a_{j_0-1},c_{j_1},a_{j_0+1},\cdots,a_n)|_{t=t_1} \ne 0$, then by Cramer criterion, we have that

$$\lim_{t\to t_1}|\partial_{x_{j_0}}u_{j_1}| = \left|\det(a_1,\cdots,a_{j_0-1},c_{j_1},a_{j_0+1},\cdots,a_n)\det(a_1,\cdots,\cdots,a_n)\right|_{t=t_1} = \infty.$$

In view of the above argument we have the following theorem:

**Theorem 2.5** *Assume that the eigenvalues of the matrix*

$$D = \partial(f_1(U_0(x_0)),\cdots,f_n(U_0(x_0)))\partial(u_1^0,\cdots,u_m^0)\cdot\partial(u_1^0,\cdots,u_m^0)\partial(x_1^0,\cdots,x_n^0)$$

are $\{\lambda_i(x_0), i = 1,\cdots,n\}$.

(1) If $\forall x_0 \in \mathbb{R}^n$, $D$ has no negative eigenvalues, then the first-order partial derivatives of solutions to equation (6) do not blow up.

(2) If there exist some $i_0$, $1 \le i_0 \le n$ and some $x_0 \in \mathbb{R}^n$, such that $\lambda_{i_0}(x_0) < 0$, denoting $t_1 = -(\lambda_{i_0}(x_0))^{-1}$, then $\det A|_{t=t_1} = 0$. Moreover if there exist some $j_0, j_1$, $1 \le j_0 \le n, 1 \le j_1 \le m$, satisfying

$$\det(a_1,\cdots,a_{j_0-1},c_{j_1},a_{j_0+1},\cdots,a_n)|_{t=t_1} \ne 0,$$

then $\lim_{t\to t_1}|\partial_{x_{j_0}}u_{j_1}| = \infty.$

Next we considering the following problem:

$$\begin{cases} U_t + \sum_{i=1}^{n} f_i(U)\partial_{x_i}U = -\alpha U,\ x \in \mathbb{R}^n,\ t > 0, \\ U(x,t)|_{t=0} = U_0(x),\ x \in \mathbb{R}^n, \end{cases} \tag{7}$$

where $\alpha \ne 0$ is a constant, $f_i \in C^\infty(\mathbb{R}^m), i = 1,\cdots,n.$ By consider the ordinary differential system:

$$\begin{cases} dx_i dt = f_i(U(x(t),t)),\ i = 1,2,\cdots,n, \\ x(0) = x_0, \end{cases}$$

we get that $U(x(t),t) = U_0(x_0)e^{-\alpha t}$, it yields that

$$x_i(t) = x_i^0 + \int_0^t f_i(U_0(x_0)e^{-\alpha\tau})d\tau,\ i = 1,\cdots,n.$$

Let

$$A^T = \partial x \partial x_0 = \partial(x_1, \cdots, x_n)\partial(x_1^0, \cdots, x_n^0),$$

$$b_j^T = \partial u_j \partial x = (\partial_{x_1} u_j, \cdots, \partial_{x_n} u_j),$$

$$c_j^T = \partial u_j^0 \partial x_0 = (\partial_{x_1^0} u_j^0, \cdots, \partial_{x_n^0} u_j^0),\ j = 1,2,\cdots, m.$$

Then

$$A^T = I + D(t) = I + \beta(t)\partial U_0(x_0)\partial x_0 \stackrel{\Delta}{=} (a_1, \cdots, a_n)^T,$$

$$Ab_j = c_j e^{-\alpha t},\ j = 1, \cdots, m,$$

where

$$\beta(t) = (\beta_{ik}(t))_{n\times m},\ \partial U_0 \partial x_0 = (\partial u_k^0 \partial x_j^0)_{m\times n},$$

$$\beta_{ik}(t) = \int_0^t \partial f_i(U_0(x_0)e^{-\alpha\tau})\partial u_k^0 d\tau,$$

$$D(t) = \beta(t)\partial U_0(x_0)\partial x_0.$$

By the similar computation, we have the following theorem:

**Theorem 2.6** *Assume that*

$$\beta_{ik}(t) = \int_0^t \partial f_i(U_0(x_0)e^{-\alpha\tau})\partial u_k^0 d\tau,$$

$$\beta(t) = (\beta_{ik}(t))_{n\times m},\ \partial U_0 \partial x_0 = (\partial u_k^0 \partial x_j^0)_{m\times n},$$

$$D(t) = \beta(t)\partial U_0(x_0)\partial x_0,$$

the eigenvalues of $D(t)$ are $\{\lambda_i(t, x_0), i = 1, \cdots, n\}$, then

(1) If $\forall t > 0$, $\forall x_0 \in \mathbb{R}^n$, $1 + \lambda_i(t, x_0) > 0, i = 1, \cdots, n$, then the first-order partial derivatives of solutions to equation (7) do not blow up.

(2) If there exist some $i_0$, $1 \le i_0 \le n$, and some $t_1 > 0$, some $x_0 \in \mathbb{R}^n$ such that $1 + \lambda_{i_0}(t_1, x_0) = 0$, moreover, if there also exist some $j_0, j_1$, $1 \le j_0 \le n, 1 \le j_1 \le m$, satisfying

$$\det(a_1, \cdots, a_{j_0 - 1}, c_{j_1} e^{-\alpha t}, a_{j_0 + 1}, \cdots, a_n)|_{t = t_1} \ne 0,$$

then $\lim_{t \to t_1} |\partial_{x_{j_0}} u_{j_1}| = \infty.$

Especially, in Theorem 2.7, if $f_i(U) = u_i, i = 1, \cdots, n$, then the corresponding system is

$$\begin{cases} U_t + \sum_{i=1}^n u_i \partial_{x_i} U = -\alpha U,\ x \in \mathbb{R}^n,\ t > 0, \\ U(x,t)|_{t=0} = U_0(x),\ x \in \mathbb{R}^n, \end{cases} \tag{8}$$

where $\alpha \ne 0$ is a constant. In this situation, we have more precise results. Here

$$\beta_{ik}(t) = \int_0^t \partial f_i(U_0(x_0)e^{-\alpha\tau})\partial u_k^0 d\tau = \int_0^t \partial(u_i(x_0)e^{-\alpha\tau})\partial u_k^0 d\tau = 1 - e^{-\alpha t}\alpha\delta_{ik},$$

so $A^T = I + 1 - e^{-\alpha t}\alpha\partial U_0 \partial x_0$. If we denote the eigenvalues of $\partial U_0 \partial x_0$ as $\{\lambda_i(x_0), i = 1, \cdots, n\}$, then the eigenvalues of $A$ are $\{1 + 1 - e^{-\alpha t}\alpha\lambda_i(x_0), i = 1, \cdots, n\}$. Since

$$\forall \alpha \neq 0, t > 0, 1 - e^{-\alpha t}\alpha > 0, \left(1 - e^{-\alpha t}\alpha\right)_t' = e^{-\alpha t} > 0,$$

using the same denotation as in Theorem 2.7, we have

$$A^T = I + 1 - e^{-\alpha t}\alpha\partial U_0(x_0)\partial x_0 \overset{\Delta}{=} (a_1, \cdots, a_n)^T,$$

$$Ab_j = c_j e^{-\alpha t}, \ j = 1, \cdots, m.$$

Then the corresponding theorem is as follows,

**Theorem 2.7** *Assume that the eigenvalues of* $\partial U_0 \partial x_0$ *are* $\{\lambda_i(x_0), i = 1, \cdots, n\}$,

(1) If $\forall x_0 \in \mathbb{R}^n$, $\lambda_i(x_0) \geq -\alpha(\alpha > 0)$, or $\lambda_i(x_0) \geq 0(\alpha < 0), i = 1, \cdots, n$ holds, then the first-order partial derivatives of solutions to equation (8) do not blow up.

(2) If there exist some $x_0 \in \mathbb{R}^n$ and some $i_0$, $1 \leq i_0 \leq n$, such that $\lambda_{i_0}(x_0) < -\alpha(\alpha > 0)$, or $\lambda_{i_0}(x_0) < 0(\alpha < 0)$ holds, denoting $t_1 = -1\alpha \ln(1 + \alpha\lambda_{i_0}(x_0))$, besides, if there also exist some $j_0$, $j_1$, $1 \leq j_0 \leq n, 1 \leq j_1 \leq m$, satisfying

$$\det(a_1, \cdots, a_{j_0 - 1}, c_{j_1} e^{-\alpha t}, a_{j_0 + 1}, \cdots, a_n)|_{t = t_1} \neq 0,$$

then $\lim_{t \to t_1} |\partial_{x_{j_0}} u_{j_1}| = \infty.$

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