MULTIPLE WIENER-ITÔ INTEGRALS

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1. INTRODUCTION

In 1938, N. wiener introduced polynomial and homogeneous chaos in his study of stochastic mechanics. He defined polynomial chaos as sums of finitely many multiple integrals with respect to a Brownian motion. The polynomial chaoses of different order are not orthogonal. On the other hand, the homogeneous chaoses (which are defined in terms of polynomial chaos) of different orders are orthogonal. However, wiener did not directly define homogeneous chaos as integrals.

In 1951, K.itô introduced new multiple integrals that turn out to be exactly homogeneous chaos. The new integrals are nowadays referred to as multiple Wiener- itô integrals. They are related to the stochastic integral that K. itô introduced in 1944. We will follow the original idea of K- itô to define multiple Wiener- itô integrals. The aim of this paper is how we can define a multiple Wiener- itô integral for deterministic function f .

(a) WIENER IDEA

The process to define the integral consists of two steps. The first step is to define the integral for step functions. The second step is to approximate an $L^2\left([a,b]^2\right)$function by step functions and take the limit of corresponding integrals. Suppose *f* is a step function given by

$f=\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}1_{[t_{i-1},t_i)\times[s_{j-1},s_j)}$, where $\{a=t_{0,}\cdots,t_{n-1},t_n=b\}$ and $\{a=s_{0,}s_1,\cdots,s_{n-1},s_m=b\}$ are

partitions of [*a*, *b*]. Then define the double wiener integral of f by $(W)\int_a^b\int_a^b f(t,s)dB(t)dB(s)$

$=\sum_{i,j}a_{ij}(B(t_i)-B(t_{i-1}))(B(s_j)-B(s_{j-1}))$

Consider a simple example $f=1_{[0,1]\times[0,1]}$ obviously we have

$$(w)\int_0^1\int_0^1 1dB(t)dB(s)=B(1)^2 \tag{1.1}$$

The double wiener integral has the separation property, namely

$$(w)\int_a^b\int_a^b f(t)g(s)dB(t)dB(s)=\int_a^b f(t)dB(t)\int_a^b g(s)dB(s) \tag{1.2}$$

(b) ITô'S IDEA

There are also two steps in defining a double wiener- itô integral. The first step is to define the integral for "off-diagonal step functions". The second step is to approximate an $L^2\left([a,b]^2\right)$ function by "off-diagonal step functions" and take the limit of the corresponding integrals.

To motivate the notion of an "off-diagonal step function" and its necessity let $\Delta_n=\{t_{0,}t_1,\cdots,t_n\}$ be a partition of the interval [0, 1]. If we take the partition of the unit square $[0,1)^2$

$$[0,1)^2=\bigcup_{i,j=1}^{n}[t_{i-1},t_i)\times[t_{j-1},t_j) \tag{1.3}$$

Then we get the following Riemann sum for the integrand $f\equiv 1$

$$\sum_{i,j}^{n}\big(B(t_i)-B(t_{i-1})\big)\big(B(t_j)-B(t_{j-1})\big) = \left[\sum_{i=1}^{n}\big(B(t_i)-B(t_{i-1})\big)\right]^2 = B(1)^2$$

Which is the value of the double wiener integral in equation (1.1).

Here is the crucial idea of K. itô in the paper [8] remove the diagonal squares from $[0,1)^2$ in equation (1.3),

$$[0,1)^2 \setminus \bigcup_{i=1}^{n}[t_{i-1},t_i)^2 = \bigcup_{i\neq j}[t_{i-1},t_i)\times[t_{j-1},t_j) \tag{1.4}$$

Then use the remaining squares in the right-hand side of this equation to form the sum from the increments of the Brownian motion

$$s_n = \sum_{i\neq j}^{n}\big(B(t_i)-B(t_{i-1})\big)\big(B(t_j)-B(t_{j-1})\big) \tag{1.5}$$

Obviously we have

$$s_n = \sum_{i,j=1}^{n}\big(B(t_i)-B(t_{i-1})\big)\big(B(t_j)-B(t_{j-1})\big)-\sum_{i=1}^{n}\big(B(t_i)-B(t_{i-1})\big)^2$$

$$= B(1)^2 - \sum_{i=1}^{n} B(t_i)-B(t_{i-1})^2$$

$$\text{Hence} \lim_{\|\Delta_n\|\to 0} s_n = B(1)^2 - 1 \text{ in } L^2(\Omega) \tag{1.6}$$

The limit is defined to be the double wiener- itô integral of $f \equiv 1$

$$\int_0^1\int_0^1 1dB(t)dB(s) = B(1)^2 - 1 \tag{1.7}$$

Which is different from the value in equation (1.1)

Let us examine more closely the sum in equation (1.5) it is the double integral of the following “off-diagonal step function”

$$f_n = \sum_{i\neq j} 1_{[t_{i-1},t_i)\times[t_{j-1},t_j)} \tag{1.8}$$

Observe that as $\|\Delta_n\| \to 0$

$$\int_0^1\int_0^1 \left|1 - f_n(t,s)\right|^2 dtds = \sum_{i=1}^{n}(t_i - t_{i-1})^2 \leq \|\Delta_n\| \to 0 \tag{1.9}$$

Hence the function $f \equiv 1$ is approximated by a sequence $\{f_n\}$ of “off-diagonal step functions” that this is a crucial idea for multiple wiener itô integral.

2. DOUBLE WIENER- ITô INTEGRALS

The object in this section is to define the double wiener- itô integral

$$\int_a^b\int_a^b f(t,s)\,dB(t)dB(s), \qquad f \in L^2\left([a,b]^2\right)$$

This will be done in two steps. However there is the new crucial notion of “off-diagonal step functions” mentioned at the end of the previous section

Let $D = \{(t,s)\in[a,b]^2; t = s\}$ denote the diagonal of the square $[a,b]^2$. By a rectangle in this paper we will mean a subset of $[a,b]^2$ of the form $[t_1,t_2)\times[s_1,s_2)$

Step 1 off-diagonal step functions

Definition 2.1 An off-diagonal step function on the square $[a,b]^2$ is defined to be a function of the form

$$f = \sum_{i\neq j} a_{ij} 1_{[t_{i-1},t_i)\times[t_{j-1},t_j)} \tag{2.1}$$

Where $a = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = b$

Note that an off-diagonal step function vanishes on the diagonal D. Hence the function $f \equiv 1$ on $[a,b]^2$ is not an off-diagonal step function. If $A = [t_1, t_2) \times [s_1, s_2)$ is a rectangle disjoint from the diagonal D then 1_A can be written in the form of (2.1) by taking the set $\{t_1, t_2, s_1, s_2\}$ as the partitions of [*a*, *b*]. Hence 1_A is an off-diagonal step function. More generally suppose $A_1, A_2, \cdots A_n$ are rectangles disjoint from the diagonal D then the function $f = \sum_{i=1}^{n} a_i 1_{A_i}$ is an off-diagonal step function for any $a_1, a_2, \cdots a_n \in \boldsymbol{R}$ this fact implies that if *f* and *g* are off-diagonal step functions, then *af* +*bg* is an off-diagonal step function for any $a, b \in R$. Hence the set of an off-diagonal step function is a vector space. For an off-diagonal step function *f* given by equation (2.1) define

$$I_2(f) = \sum_{i \neq j} a_{ij} \big(B(t_i) - B(t_{i-1})\big)\big(B(t_j) - B(t_{j-1})\big) \tag{2.2}$$

Note that the representation of an off-diagonal step function *f* by equation (2.1) is not unique, But it is easily seen that $I_2(f)$ is uniquely. The symmetrization $\hat{f}(t,s)$ of a function $f(t,s)$ is defined by

$$\hat{f}(t,s) = \frac{1}{2}\big(f(t,s) + f(s,t)\big)$$

obviously $\hat{f}$ is a symmetric function. I *f* is a symmetric function, then $\hat{f} = f$.

In general $\|\hat{f}\| \leq \|f\|$ the strict Inequality can happen. For example, for the function $f(t,s) = t$ on $[0,1]^2$, we have $\hat{f}(t,s) = \frac{1}{2}(t+s)$ and

$$\|f\|^2 = \int_0^1\int_0^1 f(t,s)^2\, dtds = \frac{1}{3},$$

$$\|\hat{f}\|^2 = \int_0^1\int_0^1 \hat{f}(t,s)^2\, dtds = \frac{7}{24}$$

Notice that the symmetrization operation is linear, i.e $(\widehat{af + bg}) = a\hat{f} + b\hat{g}$.

if *f* is an off-diagonal step function, then $\hat{f}$ is also an off-diagonal step function.

Lemma 2.2 let *f* be an off-diagonal step function then $I_2(f) = I_2(\hat{f})$.

Proof Since I_2 and the symmetrization operation are linear it suffices to prove the lemma for the case $f = 1_{[t_1, t_2) \times [s_1, s_2)}$ With $[t_1, t_2) \cap [s_1, s_2) = \phi$ the symmetrization $\hat{f}$ of *f* is given by $\hat{f} = \frac{1}{2}\left(1_{[t_1, t_2) \times [s_1, s_2)} + 1_{[s_1, s_2) \times [t_1, t_2)}\right)$

Hence by the definition of I_2 in equation (2.2),

$$I_2(f) = \big(B(t_2) - B(t_1)\big)\big(B(s_2) - B(s_1)\big)$$

$$I_2(\hat{f}) = \frac{1}{2}\big(B(t_2) - B(t_1)\big)\big(B(s_2) - B(s_1)\big) + \big(B(s_2) - B(s_1)\big)\big(B(t_2) - B(t_1)\big)$$

$$= \big(B(t_2) - B(t_1)\big)\big(B(s_2) - B(s_1)\big)$$

There for, $I_2(f) = I_2(\hat{f})$ and the lemma is proved □

Lemma 2.3 If *f* is an off-diagonal step function then $E[I_2(f)] = 0$ and

$$E\left[I_2(f)^2\right] = 2\int_a^b\int_a^b \hat{f}(t,s)^2\, dtds \tag{2.3}$$

Proof Suppose *f* is a represented by equation (2.1) then $I_2(f)$ is given by equation (2.2). Since the intervals

$[t_{i-1}, t_i)$ and $[t_{j-1}, t_j)$ are disjoint when $i \neq j$ the expectation of each term in the summation of equation (2.2) is zero hence $E[I_2(f)] = 0$

To prove equation (2.3), first assume that *f* is a symmetric. In this case $a_{ij} = a_{ji}$ for all $i \neq j$. For convenience let $\eta_i = B(t_i) - B(t_{i-1})$. then

$$I_2(f) = \sum_{i \neq j} a_{ij}\eta_i\eta_j = 2\sum_{i<j} a_{ij}\eta_i\eta_j$$

Therefore, we have

$$E\left[I_2(f)^2\right] = 4\sum_{i<j}\sum_{p<q} a_{ij}a_{pq}E\left[\eta_i\eta_j\eta_p\eta_q\right] \tag{2.4}$$

Let $i < j$ be fixed. By observing the position of intervals we can easily see the following implications:

$$p \neq i \Rightarrow E\left[\eta_i\eta_j\eta_p\eta_q\right] = 0 \qquad \forall q > p$$
$$q \neq j \Rightarrow E\left[\eta_i\eta_j\eta_p\eta_q\right] = 0 \qquad \forall p < q$$

Hence for fixed $i < j$, the summation over $p < q$ in equation (2.4) reduces to only one term given by $p = i$ and $q = j$ therefore

$$\begin{aligned} E\left[I_2(f)^2\right] &= 4\sum_{i<j} a_{ij}^{\ 2} E\left[\eta_i^{\ 2}\eta_j^{\ 2}\right] \\ &= 4\sum_{i<j} a_{ij}^{\ 2}(t_i - t_{i-1})(t_j - t_{j-1}) \\ &= 2\sum_{i \neq j} a_{ij}^{\ 2}(t_i - t_{i-1})(t_j - t_{j-1}) \\ &= 2\int_a^b\int_a^b f(t,s)^2\, dtds\,. \end{aligned}$$

Finally for any off-diagonal step function f we have $I_2(f) = I_2(\hat{f})$ By lemma 2.2.

Hence

$$E\left[I_2(f)^2\right] = E\left[I_2(\hat{f})^2\right] = 2\int_a^b\int_a^b \hat{f}(t,s)\,dtds$$

Which proves equation (2.3)

Step2 Approximation by off-diagonal step functions.

Recall that $\|f\|$ denotes the L^2-norm of a function *f* defined on the square $[a,b]^2$, i.e. $\|f\|^2 = \int_a^b\int_a^b f(t,s)^2\, dtds$

By lemma 2.3 $E\left[I_2(f)^2\right] = 2\left\|\hat{f}\right\|^2$ for any off-diagonal step function *f*. But $\left\|\hat{f}\right\| \leq \|f\|$. Hence $E\left[I_2(f)^2\right] \leq 2\|f\|^2$ for all off-diagonal step functions. This inequality shows that we can extend the mapping I_2 to $L^2\left([a,b]^2\right)$ provided that each function in $L^2\left([a,b]^2\right)$ can be approximated by sequence of off-diagonal step functions. Suppose *f* is a function in $L^2\left([a,b]^2\right)$. let D_δ denote the set of points in $[a,b]^2$ having a distance $< \delta$ from the diagonal D. For a given $\varepsilon > 0$, we can choose $\delta > 0$ small enough such that

$$\iint_{D_\delta} f(t,s)^2\, dtds < \frac{\varepsilon}{2} \tag{2.5}$$

On the other hand, let $D_\delta^C = [a,b]^2 \setminus D_\delta$ and consider the restriction of *f* to D_δ^C. A fact from measure theory says that there exists a function ϕ of the form $\phi = \sum_{i=1}^{n} a_i 1_{A_i}$ with rectangles $A_i \subset D_\delta^C$ for all $1 \le i \le n$ such that

$$\int\int_{D_\delta^C} |f(t,s) - \phi(t,s)|^2 dtds < \frac{\varepsilon}{2} \tag{2.6}$$

We can sum up the integrals in equations (2.5) and (2.6) to get

$$\int_a^b\int_a^b |f(t,s) - \phi(t,s)|^2 dtds < \varepsilon$$

Note that the function ϕ vanishes on the set D_δ. Hence the function ϕ is an off-diagonal step function

Lemma 2.4 Let *f* be a function in $L^2([a,b]^2)$ then there exists a sequence $\{f_n\}$ of off-diagonal step functions such that

$$\lim_{n\to\infty} \int_a^b\int_a^b |f(t,s) - f_n(t,s)|^2 dtds = 0 \tag{2.7}$$

Definition 2.5 Let $f \in L^2([a,b]^2)$ then

$$I_2(f) = \lim_{n\to\infty} I_2(f_n) \text{ in } L^2(\Omega) \tag{2.8}$$

Is called the double wiener-itô integral of *f*. We will also use $\int_a^b\int_a^b f(t,s)dB(t)dB(s)$ to denote the double Wiener-Itô integral $I_2(f)$ of *f*. It is easily seen that $I_2(f)$ is well-defined.

Theorem 2.6 let $f(t,s) \in L^2([a,b]^2)$ then we have

(1) $I_2(f) = I_2(\hat{f})$. Here $\hat{f}$ is the symmetrization of *f*

(2) $E[I_2(f)] = 0$

(3) $E[I_2(f)^2] = 2\|\hat{f}\|^2$. Here $\|\cdot\|$ is the norm on $L^2([a,b]^2)$

Theorem 2.7 let $f(t,s) \in L^2([a,b]^2)$ then

$$\int_a^b\int_a^b f(t,s)dB(t)dB(s) = 2\int_a^b \left[\int_a^b \hat{f}(t,s)dB(s)\right]dB(t) \tag{2.9}$$

Where $\hat{f}$ is the symmetrization of *f*.

Remark 2.8 The inner integral $X_t = \int_a^b \hat{f}(t,s)dB(s)$ is a wiener integral. Then $E(x_t^2) = \int_a^b \hat{f}(t,s)^2 ds$

Hence

$$\int_a^b E(x_t^2)dt = \int_a^b \left[\int_a^b \hat{f}(t,s)^2 ds\right]dt = \frac{1}{2}\|\hat{f}\|^2 \le \frac{1}{2}\|f\|^2 < \infty$$

This show that the stochastic process X_t belongs to $L^2_{ad}([a,b]\times\Omega)$ and the integral $\int_a^b X_t dB(t)$ is an itô integral.

Proof: first consider the case $f = 1_{[t_1,t_2)\times[s_1,s_2)}$ with $[t_1,t_2) \cap [s_1,s_2) = \phi$. The symmetrization of *f* is given by

$$\hat{f} = \frac{1}{2}\left(1_{[t_1,t_2)\times[s_1,s_2)} + 1_{[s_1,s_2)\times[t_1,t_2)}\right)$$

We may assume that $s_1 < s_2 < t_1 < t_2$ for the other case $t_1 < t_2 < s_1 < s_2$. Just interchange $[s_1,s_2)$ with $[t_1,t_2)$. By the definition of $I_2(f)$ in step 1

$$\int_a^b\int_a^b f(t,s)dB(t)dB(s) = (B(t_2) - B(t_1))(B(s_2) - B(s_1)) \tag{2.10}$$

On the other hand, we have

$$\int_a^b\left[\int_a^t \hat{f}(t,s)dB(s)\right]dB(t)=\frac{1}{2}\int_{t_1}^{t_2}\left[\int_{s_1}^{s_2}1dB(s)\right]dB(t)$$

$$=\frac{1}{2}\int_{t_1}^{t_2}\left(B(s_2)-B(s_1)\right)dB(t)=\frac{1}{2}\left(B(s_2)-B(s_1)\right)\left(B(t_2)-B(t_1)\right) \qquad (2.11)$$

Hence equation (2.9) follows from equations (2.10) and (2.11)

Next by the linearity of the mapping I_2 and the symmetrization operation, equation (2.9) is also valid for any off-diagonal step function. Finally use the approximation to extend equation (2.9) to functions $f \in L^2\left([a,b]^2\right)$

□

3. HERMITE POLYNOMIALS

The Hermite polynomial will be used to represent the multiple wiener .let *v* be the Gaussian measure with mean o and variance $\rho > 0$ i.e.

$$dv(x)=\frac{1}{\sqrt{2\pi\rho}}e^{-\frac{1}{2\rho}x^2}dx$$

Definition 3.1 The Hermite plynomial $H_n(x,\rho)$ of degree $n \in N$ and parameter *ρ*>0 is defined with

$$H_0(x,\rho)=1, \quad H_1(x,\rho)=x, \quad H_2(x,\rho)=x^2-\rho$$

And more generally from the recurrence relation

$$H_{n+1}(x,\rho)=xH_n(x,\rho)-npH_{n-1}(x,\rho)$$

In particular we have $H_n\left(x,0\right)=x^n, n\in \boldsymbol{N}$

Consider the sequence $1,x,x^2,\cdots,x^n,\cdots$ of monomials in the real Hilbert space $L^2(v)$. Apply the Gram-schmidt orthogonalization procedure to this sequence (in order of increasing powers) to obtain orthogonal polynomials $p_0(x),p_1(x),\cdots,p_n(x),\cdots$ in the Hilbert space $L^2(v)$, where $p_0(x)=1$ and $p_n(x)$ is a polynomial of degree $n\geq 1$ with leading Coefficient 1

Take function $\theta(t,x)=e^{tx}$. The expectation of $\theta(t,.)$ with respect to the measure v is given by

$$E_v\left[\theta(t,.)\right]=\int_{-\infty}^{\infty}e^{tx}\frac{1}{\sqrt{2\pi\rho}}e^{-\frac{1}{2\rho}x^2}dx=e^{\frac{1}{2}\rho t^2} \qquad (3.1)$$

The multiplicative renormalized $\Psi(t,x)$ of $\theta(t,x)$ is defined by

$$\Psi(t,x)=\frac{\theta(t,x)}{E_v\left[\theta(t,.)\right]}=e^{tx-\frac{1}{2}pt^2}$$

We can expand the function $\psi(t,x)$ as a power series in *t*.

Theorem 3.2 Let *v* be the Gaussian measure with mean *o* and variance *ρ*. Then the Hermite polynomials $H_n(x,\rho), n\geq 0,$ are orthogonal in $L^2(v)$.Moreover we have

$$\int_{-\infty}^{\infty}H_n(x,\rho)^2dv(x)=n!\rho^n \ ,n\geq 0 \qquad (3.2)$$

Proof: For any $t,s\in \mathrm{R}$ use equation

$$e^{tx-\frac{1}{2}\rho t^2}=\sum_{n=0}^{\infty}\frac{H_n(x,\rho)}{n!}t^n \text{ to get } e^{(t+s)x-\frac{1}{2}\rho(t^2+s^2)}=\sum_{n,m=0}^{\infty}\frac{t^ns^m}{n!m!}H_n(x,\rho)H_m(x,\rho) \qquad (3.3)$$

By equation (3.1)

$$\int_{-\infty}^{\infty}e^{(t+s)x-\frac{1}{2}\rho(t^2+s^2)}dv(x)=e^{-\frac{1}{2}\rho(t^2+s^2)}e^{\frac{1}{2}\rho(t+s)^2}=e^{\rho ts}$$

Therefore, upon integrating both sides of equation (3.3) we obtain

$$e^{\rho ts} = \sum_{n,\,m=0}^{\infty} \frac{t^n s^m}{n!m!} \int_{-\infty}^{\infty} H_n(x,\rho) H_m(x,\rho) dv(x) \tag{3.4}$$

Since the left-hand side is a function of the product *ts*, the coefficient of $t^n s^m$ in the right-hand side must be 0 for any $n \neq m$ namely

$$\int_{-\infty}^{\infty} H_n(x,\rho) H_m(x,\rho) dv(x) = 0 \ \forall n \neq m \tag{3.5}$$

Hence the Hermite polynomials are orthogonal in $L^2(v)$. And the equation (3.4) becomes

$$e^{\rho ts} = \sum_{n=0}^{\infty} \frac{(ts)^n}{(n!)^2} \int_{-\infty}^{\infty} H_n(x,\rho)^2 dv(x)$$

But we also have the power series expansion

$$e^{\rho ts} = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} (ts)^n$$

Obviously, we get equation (3.2) ubon comparing the coefficient of $(ts)^n$ in the last two power series.

Theorem 3.3 Let *v* be the Gaussian measure with mean 0 and variance ρ then every function *f* in $L^2(v)$ has a unique series expansion

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{H_n(x,\rho)}{\sqrt{n!\rho^n}} \tag{3.6}$$

Where the coefficients a_n are given by

$$a_n = \frac{1}{\sqrt{n!\rho^n}} \int_{-\infty}^{\infty} f(x) H_n(x,\rho) dv(x), n \geq 0$$

Moreover we have $\|f\|^2 = \sum_{n=0}^{\infty} a_n^{\ 2}$

4. HOMOGENEOUS CHAOS

Let *v* be the Gaussian measure on ℝ with mean 0 and variance ρ. By theorem 3.3 every function in the Hilbert space $L^2(v)$ has a unique expansion by the Hermite polynomials $H_n(x,\rho), n \geq 0$.

Let *C* denote the Banach space of real-valued continuous functions on $[0,1]$ vanishing at 0 and let μ be the wiener measure on *C*. the wiener space (C,μ) is an infinite dimensional analogue of the one-dimensional probability space (R, v)

Lemma 4.1 let (Ω, ς, P) be a probability space and let $\{\varsigma_n\}$ be the filtration such that $\sigma\{U_n \varsigma_n\} = \varsigma$. Suppose $X \in L^1(\Omega)$. Then $E[X \mid \varsigma_n]$ converges to *X* in $L^1(\Omega)$ as $n \to \infty$

Remark 4.2 Actually it is also true that $E[X \mid \varsigma_n]$ converges to *X* almost surely. In general if $\{X_n\}$ is an $L^1(\Omega)$-bounded martingale then X_n converges almost surely to some random variable.

Proof let $\varepsilon > 0$ be given. Since $\sigma\{U_n \varsigma_n\} = \varsigma$ there exists $S_\varepsilon = \sum_{i=1}^{k} a_i 1_{A_i}, a_i \in R \quad A_i \in U_n \varsigma_n$, such that

$$\|X - S_\varepsilon\|_1 < \varepsilon/2 \tag{4.1}$$

Where $\|\cdot\|_1$ is the norm on $L^1(\Omega)$.

For simplicity, let $X_n = E[X \mid \varsigma_n]$ then

$$\|X - X_n\|_1 = \left\|\{X - S_\varepsilon\} + \{S_\varepsilon - E[S_\varepsilon \mid \varsigma_n]\} + \{E[S_\varepsilon - X \mid \varsigma_n]\}\right\|_1$$
$$\leq \|X - S_\varepsilon\|_1 + \|S_\varepsilon - E[S_\varepsilon \mid \varsigma_n]\|_1 + \|E[S_\varepsilon - X \mid \varsigma_n]\|_1 \qquad (4.2)$$

Note that $\left|E[S_\varepsilon - X \mid \varsigma_n]\right| \leq E\left[|S_\varepsilon - X| \varsigma_n\right]$ by the conditional Jensen's inequality. Then take the expectation to get

$$\|E[S_\varepsilon - X \mid \varsigma_n]\|_1 \leq E|S_\varepsilon - X| = \|S_\varepsilon - X\|_1 < \varepsilon/2 \qquad (4.3)$$

Moreover, observe that $A_i \in U_n\varsigma_n$ for $i = 1, 2, \cdots, k$ Hence there exists N such that $A_i \in \varsigma_n$ for all $i = 1, 2, \cdots, k$ this implies that $E[S_\varepsilon \mid \varsigma_n] = S_\varepsilon, \forall n \geq N$ (4.4)

By putting equations (4.1), (4.3) and (4.4) into equation (4.2) we immediately see that $\|X - X_n\|_1 \leq \varepsilon$ for all $n \geq N$. Hence X_n converges to X in $L^1(\Omega)$ as $n \to \infty$

Now we return to the Brownian motion $B(t)$ on the probability space $(\Omega, \boldsymbol{F}^{B}, P)$. Let $I(f) = \int_a^b f(t)dB(t)$ be the wiener integral of $f \in L^2[a,b]$. A product $I(f_1)I(f_2)\cdots I(f_k)$ with $f_1, f_2, \cdots, f_k \in L^2[a,b]$ is called apolynomial chaos of order *K*.

Let $j_0 = R$ and for $n \geq 1$, define j_n to be the $L_B^2(\Omega)$- closure of the linear space spanned by constant function and polynomial chaos of degree $\leq n$ then we have the inclusions

$$j_0 \subset j_1 \subset \cdots \subset j_n \subset \cdots \subset L_B^2(\Omega) \qquad (4.5)$$

Theorem 4.3 The union $\bigcup_{n=0}^{\infty} j_n$ is dense in $L_B^2(\Omega)$

Proof Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for $L^2[a, b]$ and let ς_n be the σ-field generated by $I(e_1), I(e_2), \cdots, I(e_n)$ then $\{\varsigma_n\}$ is a filtration and we have $\sigma\{U_n\varsigma_n\} = \boldsymbol{F}^{B}$. Let $\phi \in L_B^2(\Omega)$ be orthogonal to $\bigcup_{n=0}^{\infty} j_n$, then for any fixed *n*

$$E\{\phi \cdot I(e_1)^{k_1} I(e_2)^{k_2} \cdots I(e_n)^{k_n}\} = 0 \quad \forall k_1, k_2, \cdots, k_n \geq 0$$

Observe that $I(e_1)^{k_1} I(e_2)^{k_2} \cdots I(e_n)^{k_n}$ is ς_n -measurable.

Hence $E\{\phi \cdot I(e_1)^{k_1} I(e_2)^{k_2} \cdots I(e_n)^{k_n}\} = E\{I(e_1)^{k_1} I(e_2)^{k_2} \cdots I(e_n)^{k_n} E[\phi \mid \varsigma_n]\}$

Therefore, for all integers $k_1, k_2, \cdots, k_n \geq 0$,

$$E\{I(e_1)^{k_1} I(e_2)^{k_2} \cdots I(e_n)^{k_n} E[\phi \mid \varsigma_n]\} = 0$$

Note that the random variables $I(e_1), I(e_2), \cdots, I(e_n)$ are independent with the same standard normal distribution. Moreover

$$E[\phi \mid \varsigma_n] = \theta_n(I(e_1), I(e_2), \cdots, I(e_n)) \qquad (4.6)$$

For some measurable function θ_n on R^n thus for all $k_1, k_2, \cdots k_n \geq 0$

$$\int_{R^n} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n} \theta_n(x_1, x_2, \cdots, x_n) d\mu(x) = 0$$

Where μ is the standard Gaussian measure on R^n. This equality implies that for any integers $k_1, k_2, \cdots k_n \geq 0$

$$\int_{R^n} H_{k_1}(x_1, 1) H_{k_2}(x_2, 1) \cdots H_{k_n}(x_n, 1) \theta_n(x_1, \cdots, x_n) d\mu(x) = 0$$

But it follows from theorem 3.3 that the collection

$\{H_{k_1}(x_1,1)H_{k_2}(x_2,1)\cdots H_{k_n}(x_n,1), k_1,k_2,\cdots,k_n \geq 0\}$ Is an orthogonal basis for $L^2\left(R^n,\mu\right)$, Hence $\theta_n = 0$ almost everywhere with respect to μ. Then by equation (4.6) we have

$E\left[\phi \mid \varsigma_n\right] = 0$, almost surely for any $n \geq 1$.

On other hand by lemma 4.1 $E\left[\phi \mid \varsigma_n\right]$ converges to ϕ in $L_B^2(\Omega)$ as $n \to \infty$. Hence $\phi = 0$ almost surely. This proves the assertion that the union $\bigcup_{n=0}^{\infty} j_n$ is dense in $L_B^2(\Omega)$ □

Definition 4.4 let *n* be a nonnegative integer. The elements of the Hilbert space k_n are called homogeneous chaoses of order *n*.

The space $k_n, n \geq 1$, are all infinite-dimensional. The homogeneous chaoses of order 1 are Gaussian random variables.

Theorem 4.5 the space $L_B^2(\Omega)$ is the orthogonal direct sum of the spaces k_n of homogeneous chaoses of order $n \geq 0$, namely

$L_B^2(\Omega) = k_0 \oplus k_1 \oplus k_2 \oplus \cdots \oplus k_n \oplus \cdots$

Each function ϕ in $L_B^2(\Omega)$ has a unique homogeneous chaos expansion

$$\phi = \sum_{n=0}^{\infty} \phi_n \tag{4.7}$$

Moreover $$\|\phi\|^2 = \sum_{n=0}^{\infty} \| \phi_n \|^2 \tag{4.8}$$

Where $\|.\|$ is the norm on $L_B^2(\Omega)$

Theorem 4.6 Let p_n denote the orthogonal projection of $L_B^2(\Omega)$ onto k_n, if $f_1,\cdots,f_k$ are nonzero orthogonal functions in $L^2[a,b]$ and $n_1,\cdots,n_k$ are nonnegative integers, then $p_n\left(I(f_1)^{n_1}\cdots I(f_k)^{n_k}\right) = H_{n_1}\left(I(f_1);\|f_1\|^2\right)\cdots H_{n_k}\left(I(f_k);\|f_k\|^2\right)$

Where $n = n_1 + \cdots + n_k$. In particular we have

$$p_n\left(I(f_n)\right) = H_n\left(I(f);\|f\|^2\right) \tag{4.9}$$

For any nonzero function *f* in $L^2[a,b]$

Remark 4.7 It follows from the theorem that $H_n\left(I(f);\|f\|^2\right)$ and more generally $H_{n_1}\left(I(f_1);\|f_1\|^2\right)\cdots H_{n_k}\left(I(f_k);\|f_k\|^2\right)$ with $n_1 + \cdots + n_k = n$ are all homogeneous chaoses of order *n*.

5. ORTHONOMAL BASIS FOR HOMOGENEOUS CHAOS

Let $f \in L^2[a,b]$. For convenience, we will also use the notation $\widetilde{f}$ to denote the wiener integral of *f*, namely $\widetilde{f} = I(f) = \int_a^b f(t)dB(t)$.

In this section we fix an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ for the space $L^2[a,b]$ for a sequence $\{n_k\}_{k=1}^{\infty}$ of nonnegative integers with finite sum, define

$$H_{n_1,n_2,\cdots} = \prod_k \frac{1}{\sqrt{n_k!}} H_{n_k}\left(\tilde{e}_k;1\right) = \frac{1}{\sqrt{n_1!n_2!\cdots}} H_{n_1}(\tilde{e}_1;1)H_{n_2}(\tilde{e}_2;1)\cdots \tag{5.1}$$

Note that $H_0(x;1)=1$ and there are only finitely many non zero $n_k{}'s$.
Hence the product in this equation is a product of only finitely many factors.
Lemma 5.1 For any fixed integer $n \geq 1$, the collection of functions

$$\{H_{n_1, n_2, \cdots}; n_1 + \cdots = n\}$$

Is a subset of k_n . Moreover, the linear space spanned by this collection of functions is dense in k_n .
Proof the first assertion follows from theorem 4.6. To prove the second assertion, notice that the same arguments as those in the proof of theorem 4.3 show the following implication

$$E\left[\phi H_{n_1, n_2, \cdots}\right] = 0, \forall n_1, n_2, \cdots \geq 0 \Rightarrow \phi = 0 \tag{5.2}$$

Now suppose $\phi \in k_n$ is orthogonal to $H_{n_1, n_2, \cdots}$ for all $n_1, n_2, \cdots$ with $n_1 + n_2 + \cdots = n$ then $p_n \phi = \phi$ and $E\left(\phi \cdot H_{n_1 n_2, \cdots}\right) = 0$ for all such $n_k{}'s$.

Here p_n is the orthogonal projection of $L_B^2(\Omega)$ onto k_n . Moreover, observe that for any $n_1, n_2, \cdots$

$$p_n\left(H_{n_1, n_2, \cdots}\right) = \begin{cases} H_{n_1, n_2, \cdots} & if\ n_1 + n_2 + \cdots = n; \\ 0 & otherwise \end{cases} \tag{5.3}$$

Then for any $n_1, n_2, \cdots \geq 0$, we can use the assumption on ϕ and equation (5.3) to show that

$$E\left(\phi \cdot \boldsymbol{H}_{n_1 n_2, \cdots}\right) = \left\langle \phi, H_{n_1, n_2, \cdots} \right\rangle = \left\langle p_n \phi, H_{n_1, n_2, \cdots} \right\rangle = \left\langle \phi, p_n H_{n_1, n_2, \cdots} \right\rangle$$
$$= E\left[\phi \cdot \left(p_n H_{n_1, n_2, \cdots}\right)\right] = 0$$

Where $\langle \cdot , \cdot \rangle$ is the inner product on the Hilbert space $L_B^2(\Omega)$. Therefore, by the implication in equation (5.2), we conclude that $\phi = 0$. This finishes the proof for the lemma
Theorem 5.2 For any fixed integer $n \geq 1$, the collection of functions

$$\{H_{n_1, n_2, \cdots}; n_1 + n_2 + \cdots = n\} \tag{5.4}$$

Is an orthonormal basis for the space k_n of homogeneous chaos of order n .
Theorem 5.3 the collection of functions

$$\{H_{n_1, n_2, \cdots}; n_1 + n_2 + \cdots = n, n = 0, 1, 2, \cdots\} \tag{5.5}$$

Is an orthonormal basis for the Hilbert space $L_B^2(\Omega)$. Every ϕ in $L_B^2(\Omega)$ has a unique series expansion

$$\phi = \sum_{n=0}^{\infty} \sum_{n_1 + n_2 + \cdots = n} a_{n_1, n_2, \cdots} H_{n_1, n_2, \cdots} \tag{5.6}$$

Where $a_{n_1, n_2, \cdots} = E\left(\phi H_{n_1, n_2, \cdots}\right) = \int_{\Omega} \phi H_{n_1, n_2, \cdots} dp$

6. MULTIPLE WIENER- ITô ITOINTEGRALS

Let $T = [a, b]$ the first aim in this section is to define the multiple wiener- itô integral

$$\int_{T^n} f(t_1, t_2, \cdots, t_n) dB(t_1) dB(t_2) \cdots dB(t_n) \text{ For } f \in L^2(T^n).$$

The essential idea is already given in the case $n = 2$ for the double wiener- itô integral. We will simply modify the arguments and notation in section 2 to the case $n \geq 3$.
Let $D = \{(t_1, t_2, \cdots, t_n) \in T^n; \exists i \neq j \text{ such that } t_i \neq t_j\}$ be the "diagonal set" of T^n . A subset of T^n of the form $[t_1^{(1)}, t_1^{(2)}) \times [t_2^{(1)}, t_2^{(2)}) \times \cdots \times [t_n^{(1)}, t_n^{(2)})$ is called a rectangle.

Step 1 Off-diagonal step functions

A step function on T^n is a function of the form

$$f = \sum_{1 \le i_1, i_2, \cdots, i_n \le k} a_{i_1, i_2, \cdots, i_n} 1_{[\tau_{i_1-1}, \tau_{i_1}) \times [\tau_{i_2-1}, \tau_{i_2}) \times \cdots \times [\tau_{i_n-1}, \tau_{i_n})} \quad (6.1)$$

Where $a = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_k = b$ An off-diagonal step function is a step function with coefficient satisfying the condition

$$a_{i_1, i_2, \cdots i_n} = 0 \text{ if } i_p = i_q \text{ for some } p \neq q \quad (6.2)$$

The collection of off-diagonal step functions is a vector space. For an off-diagonal step function *f* given by equation(6.1) define

$$I_n(f) = \sum_{1 \le i_1, i_2, \cdots, i_n \le k} a_{i_1, i_2, \cdots, i_n} \eta_{i_1} \eta_{i_2} \cdots \eta_{i_n} \quad (6.3)$$

Where $\eta_{i_p} = B\left(\tau_{i_p}\right) - B(\tau_{i_p - 1}), \quad 1 \le p \le n$

The $I_n(f)$ is well-defined, i.e. it does not depend on how *f* is represented by equation (6.1) Moreover, the mapping I_n is linear on the vector space of off-diagonal step functions. The symmetrization $\hat{f}(t_1, \cdots, t_n)$ of a function $f(t_1, \cdots, t_n)$ is defined by

$$\hat{f}(t_1, \cdots, t_n) = \frac{1}{n!} \sum_{\sigma} f(t_{\sigma(1)}, t_{\sigma(2)}, \cdots, t_{\sigma(n)})$$

Where the summation is over all permutations σ of the set $\{1, 2, \cdots, n\}$

Lemma 6.1 if *f* is an off-diagonal step function, then $I_n(f) = I_n(\hat{f})$

Proof: Note that I_n and the symmetrization operator are linear. Hence it suffices to prove the lemma for the case

$f = 1_{[t_1^{(1)}, t_1^{(2)}) \times [t_2^{(1)}, t_2^{(2)}) \times \cdots \times [t_n^{(1)}, t_n^{(2)})}$ Where the intervals $[t_i^{(1)}, t_i^{(2)}), 1 \le i \le n$, are disjoint. Then we have

$$I_n(f) = \prod_{i=1}^{n} \left(B(t_i^{(2)}) - B(t_i^{(1)}) \right) \quad (6.4)$$

On the other hand, the symmetrization $\hat{f}$ of *f* is given by

$\hat{f} = \frac{1}{n!} \sum_{\sigma} 1_{[t_{\sigma(1)}^{(1)}, t_{\sigma(1)}^{(2)}) \times [t_{\sigma(2)}^{(1)}, t_{\sigma(2)}^{(2)}) \times \cdots \times [t_{\sigma(n)}^{(1)}, t_{\sigma(n)}^{(2)})}$ Therefore

$I_n(\hat{f}) = \frac{1}{n!} \sum_{\sigma} \prod_{i=1}^{n} \left(B(t_{\sigma(i)}^{(2)}) - B(t_{\sigma(i)}^{(1)}) \right)$ Observe that

$\prod_{i=1}^{n} \left(B(t_{\sigma(i)}^{(2)}) - B(t_{\sigma(i)}^{(1)}) \right) = \prod_{i=1}^{n} \left(B(t_i^{(2)}) - B(t_i^{(1)}) \right)$ for any permutation σ moreover, there are $n!$ permutations of the set $\{1, 2, \cdots, \mathrm{n}\}$

It follows that

$$I_n(\hat{f}) = \frac{1}{n!} \sum_{\sigma} \prod_{i=1}^{n} \left(B(t_i^{(2)}) - B(t_i^{(1)}) \right) = \prod_{i=1}^{n} \left(B(t_i^{(2)}) - B(t_i^{(1)}) \right) \quad (6.5)$$

Equations (6.4) and (6.5) proves the assertion of the lemma

Lemma 6.2 If *f* is an off-diagonal step function, then $E[I_n(f)] = 0$ and

$$E\left[I_n(f)^2\right] = n! \int_{T^n} \left| \hat{f}(t_1, t_2, \cdots, t_n) \right|^2 dt_1 dt_2 \cdots dt_n \quad (6.6)$$

Proof: Let *f* be an off-diagonal step function given by equation (6.1) then $I_n(f)$ is given by equation (6.3). Since the function *f* satisfies the condition in equation (6.2) the coefficient $a_{i_1, i_2, i_3, \cdots, i_n}$ must be 0 whenever the intervals $[\tau_{i_1 - 1}, \tau_{i_1}), [\tau_{i_2 - 1}, \tau_{i_2}), \cdots, [\tau_{i_n - 1}, \tau_{i_n})$ are not disjoint. On the other hand when this interval are disjoint, the corresponding product $\eta_{i_1}\eta_{i_2}\cdots\eta_{i_n}$ has expectation 0. Hence we have $E\left[I_n(f)\right]=0$.

Note that $I_n(f)=I_n(\hat{f})$ by lemma 6.1.
Hence we may assume that *f* is symmetric in proving equation (6.6) in that case

$a_{i_{\sigma(1)},i_{\sigma(2)},\ldots,i_{(n)}}=a_{i_1,i_2,\cdots,i_n}$

For any permutation σ. Thus $I_n(f_n)$. In equation (6.3) can be rewritten as

$$I_n(f)=n!\sum_{1\le i_1<i_2<\cdots<i_n\le k} a_{i_1,i_2,\cdots,i_n}\eta_{i_1}\eta_{i_2}\cdots\eta_{i_n} \text{ therefore}$$

$$E\left[I_n(f)^2\right]=(n!)^2\sum_{i_1<\cdots<i_n}\sum_{j_1<\cdots<j_n} a_{i_1,i_2,\ldots,i_n}a_{j_1,j_2,\cdots,j_n}E\left[\eta_{i_1}\ldots\eta_{in}\eta_{j_1}\ldots\eta_{jn}\right]$$

Where, for simplicity of notation, we have omitted the indices as $i_2,\ j_2,$ etc. observe that for a fixed set of indices $i_1<\cdots<i_n$ we have

$$E\left[\eta_{i_1}\eta_{i_2}\cdots\eta_{in}\eta_{j_1}\ldots\eta_{jn}\right]=\begin{cases}\prod_{p=1}^{n}\left(\tau_{i_p}-\tau_{i_p-1}\right), & \text{if } j_1=i_1,\cdots,j_n=i_n;\\ 0, & \text{otherwise}\end{cases}$$

It follows that

$$E\left[I_n(f)^2\right]=(n!)^2\sum_{i_1<\cdots<i_n} a^2_{i_1,i_2,\cdots,i_n}\prod_{p=1}^{n}\left(\tau_{i_p}-\tau_{i_p-1}\right)$$

$$=n!\sum_{i_1,\cdots,i_n} a^2_{i_1,\cdots,i_n}\prod_{p=1}^{n}\left(\tau_{i_p}-\tau_{i_p-1}\right)$$

$$=n!\int_{T^n} f(t_1,\cdots,t_n)^2 dt_1\cdots dt_n$$

Which proves equation (6.6) since *f* is assumed to be symmetric

Step 2: Approximation by off-diagonal step functions.

Recall the set *D* defined earlier in this section. The set *D* can be rewritten as $D=\bigcup_{i\ne j}\left[\{t_i=t_j\}\cap D\right]$, which means that *D* is a finite union of the intersections of $(n-1)$-dimensional hyperplanes with *D*. Hence the lebesgue measure of *D* is zero.

Lemma 6.3 Let *f* be a function in $L^2(T^n)$. Then there exists a sequence $\{f_k\}$ of off-diagonal step functions such that

$$\lim_{k\to\infty}\int_{T^n}\left|f(t_1,t_2,\cdots,t_n)-f_k(t_1,t_2,\cdots,t_n)\right|^2 dt_1dt_2\cdots dt_n=0 \qquad (6.7)$$

Now suppose $f\in L^2\left(T^n\right)$. Choose a sequence $\{f_k\}$ of off-diagonal step functions converging to *f* in $L^2\left(T^n\right)$. By Lemma 6.3 such a sequence exists. Then by the linearity of I_n and lemma 6.2

$$E\left[\left(I_n(f_k)-I_n(f_L)\right)^2\right]=n!\left\|\hat{f}_k-\hat{f}_L\right\|^2\le n!\left\|f_k-f_L\right\|^2\to 0 \text{ as } k,L\to\infty.$$

Hence the sequence $\{I_n(f_k)\}_{k=1}^{\infty}$ is Cauchy in $L^2(\Omega)$

$$\text{Define } I_n(f)=\lim_{k\to\infty} I_n(f_k),\ \text{in } L^2(\Omega) \qquad (6.8)$$

The value $I_n(f)$ is well-defined, namely, it does not depend on the choise of the sequence $\{f_k\}$ used in equation (6.8).

Definition 6.4 Let $f \in L^2(T^n)$. The limit $I_n(f)$ in equation (6.8) is called the multiple Wiener- itô integral of f and is denoted by

$$\int_{T^n} f(t_1, t_2, \cdots, t_n) dB(t_1) dB(t_2) \cdots dB(t_n)$$

Note that $I_1(f)$ is simply the Wiener integral $I(f)$ of f and $I_2(f)$ is the double Wiener- itô integral of f defined in section 2.

Obviously, lemmas 6.1 and 6.2 can be extended to functions in $L^2(T^n)$

Using the approximation in lemma 6.3 and the definition of the multiple Wiener- itô integral.

Theorem 6.5 let $f \in L^2(T^n), n \geq 1$ then we have

(1) $I_n(f) = I_n(\hat{f})$. Here $\hat{f}$ is the symmerization of f

(2) $E[I_n(f)] = 0$

(3) $E[I_n(f)^2] = n! \|\hat{f}\|^2$. Here $\|\cdot\|$ is the norm on $L^2(T^n)$

The next theorem gives an equality to write a multiple Wiener- Itô integral as an iterated Itô integral. It is useful for computation.

Theorem 6.6 let $f \in L^2(T^n), n \geq 2$ then

$$\int_{T^n} f(t_1, t_2, \cdots, t_n) dB(t_1) dB(t_2) \cdots dB(t_n)$$

$$= n! \int_a^b \cdots \int_a^{t_{n-2}} \left[\int_a^{t_{n-1}} \hat{f}(t_1, t_2, \cdots, t_n) dB(t_n) \right] dB(t_{n-1}) \cdots dB(t_1)$$

Where $\hat{f}$ is the symmetrization of f

Proof It sufficices to prove the theorem for the case that f is the characteristic function of a rectangle that is disjoint from the set D. By lemma 6.1 we may assume that f is of the form $f = 1_{[t_1^{(1)}, t_1^{(2)}) \times [t_2^{(1)}, t_2^{(2)}) \times \cdots \times [t_n^{(1)}, t_n^{(2)})}$

Where $t_n^{(1)} < t_n^{(2)} \leq t_{n-1}^{(1)} < t_{n-1}^{(2)} \leq \cdots \leq t_2^{(1)} < t_2^{(2)} \leq t_1^{(1)} < t_1^{(2)}$

Then the multiple Wiener- itô integral of f is given by

$$\int_{T^n} f(t_1, t_2, \cdots, t_n) dB(t_1) dB(t_2) \cdots dB(t_n) = \prod_{i=1}^{n} \left(B\left(t_i^{(2)}\right) - B\left(t_i^{(1)}\right) \right) \tag{6.9}$$

On the other hand, note that $\hat{f} = \frac{1}{n!} f$ on the region $t_n < t_{n-1} < \cdots < t_1$.

Hence we get

$$\int_a^{t_{n-1}} \hat{f}(t_1, t_2 \cdots, t_n) dB(t_n) = \frac{1}{n!} 1_{[t_1^{(1)}, t_1^{(2)}) \times [t_2^{(1)}, t_2^{(2)}) \times \cdots \times [t_{n-1}^{(1)}, t_{n-1}^{(2)})} \left(B\left(t_n^{(2)}\right) - B\left(t_n^{(1)}\right) \right)$$ which is $\mathcal{F}_{t_{n-1}^{(1)}}$-measurable

and can be regarded as a "constant" stochastic process for integration on the interval $[t_{n-1}^{(1)}, t_{n-1}^{(2)})$ with respect to $dB(t_{n-1})$.

Hence we can repeat the above arguments to get

$$\int_a^b \cdots \int_a^{t_{n-2}} \left[\int_a^{t_{n-1}} \hat{f}(t_1, \cdots, t_n) dB(t_n) \right] dB(t_{n-1}) \cdots dB(t_1)$$

$$= \frac{1}{n!} \prod_{i=1}^{n} \left(B\left(t_i^{(2)}\right) - B\left(t_i^{(1)}\right) \right) \tag{6.10}$$

The theorem follows from equations (6.9) and (6.10)

Definition 6.7 let $g_1, \cdots, g_n \in L^2[a, b]$ the tensor product $g_1 \otimes \cdots \otimes g_n$ is defined to be the function

$$g_1 \otimes \cdots \otimes g_n (t_1, \cdots t_n) = g_1(t_1) \cdots g_n(t_n)$$

The tensor product $f_1^{\otimes n_1} \otimes \cdots \otimes f_k^{\otimes n_k}$ means that f_j is repeated n_j times, $1 \le j \le k$

Theorem 6.8 let $f_1, f_2, \cdots, f_k$ be nonzero orthogonal functions in $L^2[a,b]$ and let $n_1, \cdots n_k$ be positive integers. Then

$$I_n\left(f_1^{\otimes n_1} \otimes \cdots \otimes f_k^{\otimes n_k}\right) = \prod_{j=1}^{k} H_{n_j}\left(I(f_j); \|f_j\|^2\right) \tag{6.11}$$

Where $n = n_1 + \cdots + n_k$. In particular, for any nonzero $f \in L^2[a,b]$,

$$I_n\left(f^{\otimes n}\right) = H_n\left(I(f); \|f\|^2\right) \tag{6.12}$$

Proof. We first prove equation (6.12). The case *n*=1 is obvious. The case *n*=2 is easy to show that

$$\int_a^b \int_a^b f(t) f(s) dB(t) dB(s) = I(f)^2 - \|f\|^2$$

For general integer *n*, we use mathematical induction. Suppose equation (6.12) is valid for *n*. then by theorem 6.6

$$\int_{T^{n+1}} f(t_1) \cdots f(t_{n+1}) dB(t_1) \cdots dB(t_{n+1}) = (n+1)! \int_a^b f(t_1) X_t dB(t_1)$$

Where X_t is given by

$$X_t = \int_a^t \cdots \left[\int_a^{t_n} f(t_2) \cdots f(t_{n+1}) dB(t_{n+1}) \right] \cdots dB(t_2)$$

By theorem 6.6 and the induction on *n*

$$X_t = \frac{1}{n!} \int_{[a,t]^n} f(t_2) \cdots f(t_{n+1}) dB(t_2) \cdots dB(t_{n+1}) = \frac{1}{n!} H_n\left(\int_a^t f(s) dB(s); \int_a^t f(s)^2 ds \right)$$

Therefore, we have the equality

$$\int_{T^{n+1}} f(t_1) \cdots f(t_{n+1}) dB(t_1) \cdots dB(t_{n+1})$$

$$= (n+1) \int_a^b f(t_1) H_n\left(\int_a^{t_1} f(s) dB(s); \int_a^{t_1} f(s)^2 ds \right) dB(t_1) \tag{6.13}$$

On the other hand, we can apply itô's formula to $H_{n+1}(x, \rho)$ to get

$$dH_{n+1}\left(\int_a^t f(s) dB(s), \int_a^t f(s)^2 ds \right)$$

$$= \left(\frac{\partial}{\partial x} H_{n+1} \right) f(t) dB(t) + \frac{1}{2} \left(\frac{\partial^2}{\partial x^2} H_{n+1} \right) f(t)^2 dt + \left(\frac{\partial}{\partial \rho} H_{n+1} \right) f(t)^2 dt$$

But we have the following identities

$$\frac{\partial}{\partial x} H_{n+1}(x; \rho) = (n+1) H_n(x; \rho),$$

$$\frac{\partial}{\partial \rho} H_{n+1}(x; p) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} H_{n+1}(x; \rho)$$

Thus we obtain $dH_{n+1}\left(\int_a^t f(s) dB(s); \int_a^t f(s)^2 ds \right)$

$$= (n+1) f(t) H_n\left(\int_a^t f(s) dB(s); \int_a^t f(s)^2 ds \right) dB(t)$$

Which, upon integration over $[a,b]$, gives the equality

$$H_{n+1}\left(I(f); \|f\|^2\right) = (n+1) \int_a^b f(t) H_n\left(\int_a^t f(s) dB(s); \int_a^t f(s)^2 ds \right) dB(t) \tag{6.14}$$

Equations (6.13) and (6.14) shows that equation (6.12) is valid for $n+1$.
Hence by induction, equation (6.12) holds for any positive integer *n*.

Now we prove equation (6.11).

Let $\tilde{f} = I(f)$. For any real numbers $r_1, r_2, \cdots, r_k$ apply the equation

$$e^{tI(f)-\frac{1}{2}\|f\|^2 t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n\left(I(f); \|f\|^2\right).$$

To get $\exp\left[\sum_{i=1}^{k} r_i \tilde{f}_i - \frac{1}{2}\sum_{i=1}^{k} r_i^2 \|f_i\|^2\right] = \prod_{i=1}^{k} e^{r_i \tilde{f}_i - \frac{1}{2} r_i^2 \|f_i\|^2}$

$$= \prod_{i=1}^{k} \sum_{n_i=0}^{\infty} \frac{r_i^{n_i}}{n_i!} H_{n_i}\left(\tilde{f}_i; \|f_i\|^2\right) \tag{6.15}$$

On other hand, by the equation

$$e^{tI(f)-\frac{1}{2}\|f\|^2 t^2} = \sum_{n-0}^{\infty} \frac{t^n}{n!} H_n\left(I(f); \|f\|^2\right) \text{ With } t=1 \text{ and } f = \sum_{i=1}^{k} r_i \tilde{f}_i$$

$$\exp\left[\sum_{i=1}^{k} r_i \tilde{f}_i - \frac{1}{2}\sum_{i=1}^{k} r_i^2 \|f_i\|^2\right] = \sum_{m=0}^{\infty} \frac{1}{m!} H_m\left(\sum_{i=1}^{k} r_i \tilde{f}_i; \frac{1}{2}\sum_{i=1}^{k} r_i^2 \|f_i\|^2\right)$$

Then we apply what we have already proved in equation (6.12) to H_m in the right hand side to get

$$\exp\left[\sum_{i=1}^{k} r_i \tilde{f}_i - \frac{1}{2}\sum_{i=1}^{k} r_i^2 \|f_i\|^2\right] = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{T^m} \prod_{j=1}^{m} \left[\sum_{i=1}^{k} r_i f_i(t_j)\right] dB(t_1)\cdots dB(t_m) \tag{6.16}$$

Equation (6.11) follow by comparing the coefficient of $r_1^{n_1} r_2^{n_2} \cdots r_k^{n_k}$ in the right hand sides of equations (6.15) and (6.16) □

Theorem 6.9 Let $n \neq m$ then $E\left[I_n(f) I_m(g)\right] = 0$ for any $f \in L^2(T^n)$ and $g \in L^2(T^m)$.

Proof It suffices to prove the theorem for *f* and *g* of the following form:

$$f = 1_{[t_1^{(1)}, t_1^{(2)}) \times [t_2^{(1)}, t_2^{(2)}) \times \cdots \times [t_n^{(1)}, t_n^{(2)})}$$

$$g = 1_{[s_1^{(1)}, s_1^{(2)}) \times [s_2^{(1)}, s_2^{(2)}) \times \cdots \times [s_m^{(1)}, s_m^{(2)})}$$

Where the intervals satisfy the condition

$$t_n^{(1)} < t_n^{(2)} \le t_{n-1}^{(1)} < t_{n-1}^{(2)} \le \cdots \le t_2^{(1)} < t_2^{(2)} \le t_1^{(1)} < t_1^{(2)},$$

$$s_m^{(1)} < s_m^{(2)} \le s_{m-1}^{(1)} < s_{m-1}^{(2)} \le \cdots \le s_2^{(1)} < s_2^{(2)} \le s_1^{(1)} < s_1^{(2)}$$

Then $I_n(f) I_m(g)$ is given by

$$I_n(f) I_m(g) = \left[\prod_{i=1}^{n} \left(B(t_i^{(2)}) - B(t_i^{(1)})\right)\right] \left[\prod_{j=1}^{m} \left(B(s_j^{(2)}) - B(s_j^{(1)})\right)\right] \tag{6.17}$$

Now, put these points *t* and *s* together to form an increasing set of points $\tau_1 < \tau_2 \cdots < \tau_r$ with $r \le n+m$. Then each factor in the first product of equation (6.17) can be rewritten as a sum of increments of $B(t)$ on the τ-intervals. Hence, upon multiplying out the *n* factors, each term in the first product $\prod_{i=1}^{n}$ must be of the form

$$\left(B(\tau_{i_1}) - B\left(\tau_{i_1-1}\right)\right) \cdots \left(B(\tau_{i_n}) - B\left(\tau_{i_n-1}\right)\right) \tag{6.18}$$

Where $\tau_{i_1} < \cdots < \tau_{i_n}$. Similarly, each term in the second product $\prod_{j=1}^{m}$ in equation (6.17) must be of the form

$$\left(B(\tau_{j_1}) - B\left(\tau_{j_1-1}\right)\right) \cdots \left(B(\tau_{j_m}) - B\left(\tau_{j_m-1}\right)\right) \tag{6.19}$$

Where $\tau_{j_1} < \cdots < \tau_{j_m}$.

It is easy to see that the product of equations (6.18) and (6.19) has expectation o because $n \neq m$. Thus we can conclude from equation (6.17) that $E\left[I_n(f)I_m(g)\right]=0$

7. WIENER- ITô THEOREM

Theorem 7.1 (Wiener- itô theorem)

The space $L_B^2(\Omega)$ can be decomposed into the orthogonal direct sum

$$L_B^2(\Omega)=k_0\oplus k_1\oplus k_2\oplus\cdots\oplus k_n\oplus\cdots$$

Where k_n consists of multiple Wiener- itô integrals of order *n*. Each function ϕ in $L_B^2(\Omega)$ can be uniquely represented by

$$\phi=\sum_{n=0}^{\infty}I_n(f_n),\qquad f_n\in L_{sym}^2(T^n) \tag{7.1}$$

And the following equality holds $\|\phi\|^2=\sum_{n=0}^{\infty}n!\|f_n\|^2$

Definition 7.2 Let $\phi=I_n(f),\ f\in L_{sym}^2(T^n)$.The variational derivative of ϕ is defined to be

$$\frac{\delta}{\delta t}\phi=nI_{n-1}\left(f_n(t,\cdot)\right) \tag{7.2}$$

Where the right-hand side is understood to be 0 when $n=0$. In particular

$$\frac{\delta}{\delta t}I(f)=f(t) \tag{7.3}$$

Theorem 7.3 Let $\phi\in L_B^2(\Omega)$. Assume that all variational derivatives exist and have expectation.

Put $f_n(t_1,t_2,\cdots,t_n)=\frac{1}{n!}E\left[\frac{\delta^2}{\delta t_1\delta t_2\cdots\delta t_n}\phi\right]$ then the Wiener- itô expansion of ϕ is given by $\phi=\sum_{n=0}^{\infty}I_n(f_n)$.

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